Rational solutions of the Sasano system of type $D_5^{(1)}$

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Abstract

In this paper, we classified the rational solutions of the Sasano system of type $D_5^{(1)}$, which is given by the coupled $P_{\rm V}$ system with the affine Weyl group symmetry of type $D_5^{(1)}$. The rational solutions are classified to four types by the Bäcklund transformation group.

Introduction

In this paper, we classified the rational solutions of the Sasano system of type $D_5^{(1)}$, which is defined by

$$D_5^{(1)}(\alpha_j)_{0\le j\le 5}\begin{cases} tx' = 2x^2y + tx^2 - 2xy - \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\}\, x \\ \qquad\qquad\qquad\qquad + (\alpha_2 + \alpha_5) + 2z\{(z-1)w + \alpha_3\}, \\ ty' = -2xy^2 + y^2 - 2txy + \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\}\, y - \alpha_1 t, \\ tz' = 2z^2w + tz^2 - 2zw - \{t + (\alpha_5 + \alpha_4)\}\, z + \alpha_5 + 2yz(z-1), \\ tw' = -2zw^2 + w^2 - 2tzw + \{t + (\alpha_5 + \alpha_4)\}\, w - \alpha_3 t - 2y(-w + 2zw + \alpha_3), \\ \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 = 1, \end{cases}$$

where $' = d/dt$ and $\alpha_i\,(0 \le i \le 5)$ are all arbitrary complex parameters. $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ is expressed by the coupled $P_{\rm V}$ system:

$$t\frac{dx}{dt} = \frac{\partial H}{\partial y},\ t\frac{dy}{dt} = -\frac{\partial H}{\partial x},\ t\frac{dz}{dt} = \frac{\partial H}{\partial w},\ t\frac{dw}{dt} = -\frac{\partial H}{\partial z},$$

where the Hamiltonian H is given by

$$\begin{aligned} H &= H_{\rm V}(x,y,t;\alpha_2+\alpha_5,\alpha_1,\alpha_2+2\alpha_3+\alpha_4) + H_{\rm V}(z,w,t;\alpha_5,\alpha_3,\alpha_4) + 2yz\{(z-1)w+\alpha_3\} \\ &= x(x-1)y(y+t) - (2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)xy + (\alpha_2+\alpha_5)y + \alpha_1 tx \\ &\quad + z(z-1)w(w+t) - (\alpha_5+\alpha_4)zw + \alpha_5 w + \alpha_3 tz \\ &\quad + 2yz\{(z-1)w+\alpha_3\}, \\ H_{\rm V}(q,p,t;\gamma_1,\gamma_2,\gamma_3) &= q(q-1)p(p+t) - (\gamma_1+\gamma_3)qp + \gamma_1 p + \gamma_2 tq. \end{aligned}$$

$D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ possess the Bäcklund transformations, $s_i, \pi_j\,(0 \le i \le 5,\, 1 \le j \le 4)$, which are defined as follows:

( )	$s_0(\ )$	$s_1(\ )$	$s_2(\ )$	$s_3(\ )$	$s_4(\ )$	$s_5(\ )$	$\pi_1(\ )$	$\pi_2(\ )$	$\pi_3(\ )$	$\pi_4(\ )$
x	$x+\frac{\alpha_0}{(y+t)}$	$x+\frac{\alpha_1}{y}$	x	x	x	x	$1-x$	$\frac{y+w+t}{t}$	$1-x$	x
y	y	y	$y-\frac{\alpha_2}{(x-z)}$	y	y	y	$-y-t$	$-t(z-1)$	$-y$	$y+t$
z	z	z	z	$z+\frac{\alpha_3}{w}$	z	z	$1-z$	$\frac{y+t}{t}$	$1-z$	z
w	w	w	$w+\frac{\alpha_2}{(x-z)}$	w	$w-\frac{\alpha_4}{(z-1)}$	$w-\frac{\alpha_5}{z}$	$-w$	$-t(x-z)$	$-w$	w
t	t	t	t	t	t	t	t	$-t$	$-t$	$-t$
α_0	$-\alpha_0$	α_0	$\alpha_0+\alpha_2$	α_0	α_0	α_0	α_1	α_5	α_0	α_1
α_1	α_1	$-\alpha_1$	$\alpha_1+\alpha_2$	α_1	α_1	α_1	α_0	α_4	α_1	α_0
α_2	$\alpha_2+\alpha_0$	$\alpha_2+\alpha_1$	$-\alpha_2$	$\alpha_2+\alpha_3$	α_2	α_2	α_2	α_3	α_2	α_2
α_3	α_3	α_3	$\alpha_3+\alpha_2$	$-\alpha_3$	$\alpha_3+\alpha_4$	$\alpha_3+\alpha_5$	α_3	α_2	α_3	α_3
α_4	α_4	α_4	α_4	$\alpha_4+\alpha_3$	$-\alpha_4$	α_4	α_5	α_1	α_5	α_4
α_5	α_5	α_5	α_5	$\alpha_5+\alpha_3$	α_5	$-\alpha_5$	α_4	α_0	α_4	α_5

The Bäcklund transformation group $\langle s_i, \pi_j |\ 0 \leq i \leq 5,\ 1 \leq j \leq 4\rangle$ is isomorphic to the affine Weyl group of type $D_5^{(1)}$.

In this paper, we define the coefficients of the Laurent series of x, y, z, w at $t = \infty,\ 0,\ c \in \mathbb{C}^*$ by

$$\begin{cases} a_{\infty,k},\ a_{0,k},\ a_{c,k}\ k \in \mathbb{Z},\ (\text{for } x), & b_{\infty,k},\ b_{0,k},\ b_{c,k}\ k \in \mathbb{Z},\ (\text{for } y) \\ c_{\infty,k},\ c_{0,k},\ c_{c,k}\ k \in \mathbb{Z},\ (\text{for } z), & d_{\infty,k},\ d_{0,k},\ d_{c,k}\ k \in \mathbb{Z},\ (\text{for } w), \end{cases}$$

respectively. If $b_{\infty,1} + d_{\infty,1} \neq -1/2$, we say that (x, y, z, w) is a solution of type A. If $b_{\infty,1} + d_{\infty,1} = -1/2$, we say that (x, y, z, w) is a solution of type B. Moreover, we denote the coefficients of the Laurent series of H at $t = \infty, 0, c \in \mathbb{C}^*$ by

$$h_{\infty,k},\ h_{0,k},\ h_{c,k},\ \ (k \in \mathbb{Z}),$$

respectively.

Our main theorem is as follows:

Theorem 0.1. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a rational solution. By some Bäcklund transformations, the parameters and solution can then be transformed so that one of the following occurs:*

(a-1) $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_0, 0, 0, 0, \alpha_4, 0)$ *and* $(x, y, z, w) = (0, 0, 0, 0)$,

(a-2) $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 0, 0, 0, 1, 0)$ *and*

$$(x, y, z, w) = (0, 0, 0, 0), (0, -t, 0, 0), (0, 0, 0, -t), (0, -t, 0, t),$$

(b-1) $-\alpha_0 + \alpha_1 = -\alpha_4 + \alpha_5 = 0$ *and* $(x, y, z, w) = (1/2, -t/2, 1/2, 0)$,

(b-2) $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (1/2, 1/2, 0, \alpha_3, -\alpha_3, -\alpha_3)$ *and*

$$(x, y, z, w) = (1/2, -t/2 + b, 1/2, d),$$

where b, d are both arbitrary complex numbers and satisfy $b + d = 0$.

This paper is organized as follows. In Sections 1, 2 and 3, we treat the meromorphic solutions at $t = \infty, 0, c \in \mathbb{C}^*$ of $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$.

We first find that for a meromorphic solution of type A at $t = \infty$, $a_{\infty,-1}(= -\mathrm{Res}_{t=\infty} x)$ is given by the parameters, and for a meromorphic solution of type B at $t = \infty$, $b_{\infty,0} + d_{\infty,0} = 0, -\alpha_4 + \alpha_5$. We next see that for a meromorphic solution at $t = 0$, $b_{0,0} + d_{0,0} = 0, -\alpha_4 + \alpha_5$. We last observe that for a meromorphic solution at $t = c \in \mathbb{C}^*$, $a_{c,-1} = \mathrm{Res}_{t=c} x \in \mathbb{Z}$, and $b_{c,-1} + d_{c,-1} = \mathrm{Res}_{t=c}(y + w) = nc$ $(n \in \mathbb{Z})$. Therefore, it follows that

$$a_{\infty,-1} - a_{0,-1} \in \mathbb{Z} \text{ and } (b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}.$$

In Section 4, for a meromorphic solution at $t = \infty, 0, c \in \mathbb{C}^*$, we compute $h_{\infty,0}, h_{0,0}$, and $h_{c,-1} = \mathrm{Res}_{t=c} H$. We then see that $h_{\infty,0}, h_{0,0}$ are both expressed by the parameters and $h_{c,-1} = nc$ $(n \in \mathbb{Z}_{\ge 0})$. Therefore, it follows that

$$h_{\infty,0} - h_{0,0} \in \mathbb{Z}_{\ge 0}.$$

In Section 5, we investigate the properties of the Bäcklund transformations and show the existence of the "infinite solution," which is given by $y \equiv w \equiv \infty$ or $z \equiv \infty$. In Section 6, we determine the infinite solution.

In Sections 7 and 8, we obtain necessary conditions for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ to have rational solutions of type A and B. For this purpose, we use

$$a_{\infty,-1} - a_{0,-1} \in \mathbb{Z},\ (b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z} \text{ and } h_{\infty,0} - h_{0,0} \in \mathbb{Z}_{\ge 0},$$

and express the necessary conditions by the parameters.

In Section 9, using the Bäcklund transformations, we transform the parameters so that one of the following occurs:
(1) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = 0$, (2) $-\alpha_0 + \alpha_1 = -\alpha_4 + \alpha_5 = 0$, (3) $-\alpha_0 + \alpha_1 = 0, -\alpha_4 + \alpha_5 = 1$. In this paper, we call cases (2) and (3) the standard forms I and II.

In Sections 10, 11, we treat case (1), that is, determine the rational solutions of type A of $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$. In Section 12, we obtain the main theorem for type A.

In Sections 13, 14, 15, 16 and 17, we determine the rational solutions of type B for the standard form I. In Sections 18, 19, 20, 21 and 22, we determine the rational solutions of type B for the standard form II. In Section 23, we obtain the main theorem for type B.

Acknowledgments

The author wishes to express his sincere thanks to Professor Yousuke Ohyama.

1 Meromorphic solutions at $t = \infty$

In this section, for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, we determine the Laurent series of (x, y, z, w) at $t = \infty$. For this purpose, we consider the following five cases:
(0) all of (x, y, z, w) are holomorphic at $t = \infty$,
(1) one of (x, y, z, w) has a pole at $t = \infty$,
(2) two of (x, y, z, w) have a pole at $t = \infty$,
(3) three of (x, y, z, w) have a pole at $t = \infty$,
(4) all of (x, y, z, w) have a pole at $t = \infty$.
Moreover, we set

$$\begin{cases} x = a_{\infty,n_0} t^{n_0} + a_{\infty,n_0-1} t^{n_0-1} + \cdots + a_{\infty,-1} t^{-1} + \cdots , \\ y = b_{\infty,n_1} t^{n_1} + b_{\infty,n_1-1} t^{n_1-1} + \cdots + b_{\infty,0} + \cdots , \\ z = c_{\infty,n_2} t^{n_2} + c_{\infty,n_2-1} t^{n_2-1} + \cdots + c_{\infty,-1} t^{-1} + \cdots , \\ w = d_{\infty,n_2} t^{n_3} + d_{\infty,n_3-1} t^{n_3-1} + \cdots + d_{\infty,0} + \cdots , \end{cases}$$

where n_0, n_1, n_2, n_3 are all integers.

1.1 The case where all of (x, y, z, w) are holomorphic at $t = \infty$

In this subsection, we treat the case where all of (x, y, z, w) are holomorphic at $t = \infty$.

Proposition 1.1. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that x, y, z, w are all holomorphic at $t = \infty$. Then, $a_{\infty,0} = 0, 1$ and $c_{\infty,0} = 0, 1$.*
(1) *If $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$,*

$$\begin{cases} x = (\alpha_2 + \alpha_5) t^{-1} + \cdots , \\ y = \alpha_1 + \alpha_1(-\alpha_1 - 2\alpha_3 - \alpha_4 + \alpha_5) t^{-1} + \cdots , \\ z = \alpha_5 t^{-1} + \cdots , \\ w = \alpha_3 + \alpha_3(-\alpha_3 - \alpha_4 + \alpha_5) t^{-1} + \cdots . \end{cases}$$

(2) *If $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$,*

$$\begin{cases} x = (\alpha_2 + 2\alpha_3 + \alpha_5) t^{-1} + \cdots , \\ y = \alpha_1 + \alpha_1(-\alpha_1 + 2\alpha_3 - \alpha_4 + \alpha_5) t^{-1} + \cdots , \\ z = 1 + \alpha_4 t^{-1} + \cdots , \\ w = -\alpha_3 - \alpha_3(\alpha_3 - \alpha_4 + \alpha_5) t^{-1} + \cdots . \end{cases}$$

(3) *If* $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$,

$$\begin{cases} x = 1 + (\alpha_2 + 2\alpha_3 + \alpha_4)t^{-1} + \cdots, \\ y = -\alpha_1 - \alpha_1(\alpha_1 - 2\alpha_3 - \alpha_4 + \alpha_5)t^{-1} + \cdots, \\ z = \alpha_5 t^{-1} \cdots, \\ w = \alpha_3 + \alpha_3(-\alpha_3 - \alpha_4 + \alpha_5)t^{-1} + \cdots. \end{cases}$$

(4) *If* $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$,

$$\begin{cases} x = 1 + (\alpha_2 + \alpha_4)t^{-1} + \cdots, \\ y = -\alpha_1 - \alpha_1(\alpha_1 + 2\alpha_3 - \alpha_4 + \alpha_5)t^{-1} + \cdots, \\ z = 1 + \alpha_4 t^{-1} + \cdots, \\ w = -\alpha_3 - \alpha_3(\alpha_3 - \alpha_4 + \alpha_5)t^{-1} + \cdots. \end{cases}$$

Proof. It can be proved by direct calculation. □

Proposition 1.2. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$*, there exists a solution such that* x, y, z, w *are holomorphic at* $t = \infty$*. Moreover, assume that* $a_{\infty,0}$ *and* $c_{\infty,0}$ *are both determined. The solution is then unique.*

Proof. By Proposition 1.1, we can set

$$\begin{cases} x = a_{\infty,0} + a_{\infty,-1}t^{-1} + \cdots + a_{\infty,-k}t^{-k} + a_{\infty,-(k+1)}t^{-(k+1)} + \cdots, \\ y = b_{\infty,0} + b_{\infty,-1}t^{-1} + \cdots + b_{\infty,-k}t^{-k} + b_{\infty,-(k+1)}t^{-(k+1)} + \cdots, \\ z = c_{\infty,0} + c_{\infty,-1}t^{-1} + \cdots + c_{\infty,-k}t^{-k} + c_{\infty,-(k+1)}t^{-(k+1)} + \cdots, \\ w = d_{\infty,0} + d_{\infty,-1}t^{-1} + \cdots + d_{\infty,-k}t^{-k} + d_{\infty,-(k+1)}t^{-(k+1)} + \cdots, \end{cases}$$

where $a_{\infty,0}$, $a_{\infty,-1}$, $b_{\infty,0}$, $b_{\infty,-1}$, $c_{\infty,0}$, $c_{\infty,-1}$, $d_{\infty,0}$ and $d_{\infty,-1}$ have been already determined. We treat the case where $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. The other cases can be proved in the same way.

Comparing the coefficients of the term t^{-k} $(k \ge 1)$ in

$$tx' = 2x^2y + tx^2 - 2xy - \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\}x + (\alpha_2 + \alpha_5) + 2z\{(z-1)w + \alpha_3\},$$

we have

$$\begin{aligned} a_{\infty,-(k+1)} = &k a_{\infty,-k} + 2\sum a_{\infty,-l}a_{\infty,-m}b_{\infty,-n} + \sum a_{\infty,-l}a_{\infty,-m} \\ &- 2\sum a_{\infty,-l}b_{\infty,-m} - (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)a_{\infty,-k} \\ &+ 2\sum c_{\infty,-l}c_{\infty,-m}d_{\infty,-n} - 2\sum c_{\infty,-l}d_{\infty,-m}, \end{aligned} \tag{1.1}$$

where the first and fourth sums extend over the non-negative integers, l, m, n, such that $l+m+n=k$, and the second sum extends over the non-negative integers, l, m, such that $l+m=k+1$, and the third and fifth sums extend over the non-negative integers, l, m, n, such that $l+m=k$.

Comparing the coefficients of the term t^{-k} $(k \geq 1)$ in

$$ty' = -2xy^2 + y^2 - 2txy + \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\} y - \alpha_1 t,$$

we obtain

$$\begin{aligned} b_{\infty,-(k+1)} = &- kb_{\infty,-k} + 2\sum a_{\infty,-l}b_{\infty,-m}b_{\infty,-n} - \sum b_{\infty,-l}b_{\infty,-m} \\ &+ 2\sum a_{\infty,-l}b_{\infty,-m} - (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)b_{\infty,-k}, \end{aligned} \tag{1.2}$$

where the first sum extends over the non-negative integers, l, m, n, such that $l+m+n=k$, and the second sum extends over the non-negative integers, l, m, such that $l+m=k$, and the third sum extends over the non-negative integers, l, m, such that $l+m=k+1$.

Comparing the coefficients of the term t^{-k} $(k \geq 1)$ in

$$tz' = 2z^2w + tz^2 - 2zw - \{t + (\alpha_5 + \alpha_4)\} z + \alpha_5 + 2yz(z-1),$$

we have

$$\begin{aligned} c_{\infty,-(k+1)} = &kc_{\infty,-k} + 2\sum c_{\infty,-l}c_{\infty,-m}d_{\infty,-n} + \sum c_{\infty,-l}c_{\infty,-m} \\ &- 2\sum c_{\infty,-l}d_{\infty,-m} - (\alpha_5 + \alpha_4)c_{\infty,-k} \\ &+ 2\sum b_{\infty,-l}c_{\infty,-m}c_{\infty,-n} - 2\sum b_{\infty,-l}c_{\infty,-m}, \end{aligned} \tag{1.3}$$

where the first and fourth sums extend over the non-negative integers, l, m, n, such that $l+m+n=k$, and the second sum extends over the non-negative integers, l, m, such that $l+m=k+1$, and the third and fifth sums extend over the non-negative integers, l, m, such that $l+m=k$.

Comparing the coefficients of the term t^{-k} $(k \geq 1)$ in

$$tw' = -2zw^2 + w^2 - 2tzw + \{t + (\alpha_5 + \alpha_4)\} w - \alpha_3 t - 2y(-w + 2zw + \alpha_3),$$

we obtain

$$\begin{aligned} d_{\infty,-(k+1)} = &- kd_{\infty,-k} + 2\sum c_{\infty,-l}d_{\infty,-m}d_{\infty,-n} - \sum d_{\infty,-l}d_{\infty,-m} \\ &+ 2\sum c_{\infty,-l}d_{\infty,-m} - (\alpha_5 + \alpha_4)d_{\infty,-k} \\ &- 2\sum b_{\infty,-l}d_{\infty,-m} + 4\sum b_{\infty,-l}c_{\infty,-m}d_{\infty,-n} + 2\alpha_3 b_{\infty,-k}, \end{aligned} \tag{1.4}$$

where the first and fifth sums extend over the non-negative integers, l, m, n, such that $l+m+n=k$, and the second and fourth sums extend over the non-negative integers, l, m, such that $l+m=k$, and the third sum extends over the non-negative integers, l, m, such that $l+m=k+1$.

Equations (1.1), (1.2), (1.3), and (1.4) imply that $a_{\infty,-k}$, $b_{\infty,-k}$, $c_{\infty,-k}$, and $d_{\infty,-k}$ are inductively determined, which proves the proposition. $\square$

Corollary 1.3. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that x, y, z, w are all holomorphic at $t=\infty$. Then,*

$$\begin{cases} y \equiv 0 & \textit{if } \alpha_1 = 0, \\ w \equiv 0 & \textit{if } \alpha_3 = 0. \end{cases}$$

1.2 The case where one of (x, y, z, w) has a pole at $t=\infty$

In this subsection, we treat the case where one of (x, y, z, w) has a pole at $t=\infty$. For this purpose, we consider the following four cases:
(1) x has a pole at $t=\infty$ and y, z, w are all holomorphic at $t=\infty$,
(2) y has a pole at $t=\infty$ and x, z, w are all holomorphic at $t=\infty$,
(3) z has a pole at $t=\infty$ and x, y, w are all holomorphic at $t=\infty$,
(4) w has a pole at $t=\infty$ and x, y, z are all holomorphic at $t=\infty$.

1.2.1 The case where x has a pole at $t=\infty$

Proposition 1.4. *For $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists no solution such that x has a pole at $t=\infty$ and y, z, w are all holomorphic at $t=\infty$.*

Proof. It can be easily checked. $\square$

1.2.2 The case in which y has a pole at $t=\infty$

Lemma 1.5. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that y has a pole at $t=\infty$ and x, z, w are all holomorphic at $t=\infty$. y then has a pole of order one at $t=\infty$.*

Proof. It can be proved by direct calculation. $\square$

Proposition 1.6. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that y has a pole at $t=\infty$ and x, z, w are all holomorphic at $t=\infty$. Then, $b_{\infty,1}=-1, -1/2$.*

Proof. By comparing the coefficients of the term t in

$$tx' = 2x^2y + tx^2 - 2xy - \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\}\, x \\ + (\alpha_2 + \alpha_5) + 2z\, \{(z-1)w + \alpha_3\}\, ,$$

we have

$$a_{\infty,0}(a_{\infty,0} - 1)(2b_{\infty,1} + 1) = 0,$$

which implies that $a_{\infty,0} = 0, 1$ or $b_{\infty,1} = -1/2$.

By comparing the coefficients of the term t^2 in

$$ty' = -2xy^2 + y^2 - 2txy + \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\}\, y - \alpha_1 t,$$

we get

$$(b_{\infty,1} + 1)(-2a_{\infty,0} + 1) = 0,$$

which implies that $b_{\infty,1} = -1$ or $a_{\infty,0} = 1/2$.

Therefore, if $a_{\infty,0} = 0, 1$, it follows that $b_{\infty,1} = -1$. Furthermore, if $b_{\infty,1} = -1/2$, it follows that $a_{\infty,0} = 1/2$. $\square$

Let us first treat the case where $b_{\infty,1} = -1$.

Proposition 1.7. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $b_{\infty,1} = -1$. Then, $a_{\infty,0} = 0, 1$ and $c_{\infty,0} = 0, 1$.*

(1) *If $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$,*

$$\begin{cases} x = -(\alpha_2 + \alpha_5)t^{-1} + \cdots\, , \\ y = -t + \alpha_0 + \alpha_0(\alpha_0 + 2\alpha_3 + \alpha_4 - \alpha_5)t^{-1} + \cdots\, , \\ z = -\alpha_5 t^{-1} + \cdots\, , \\ w = \alpha_3 + \alpha_3(\alpha_3 + \alpha_4 - \alpha_5)t^{-1} + \cdots\, . \end{cases}$$

(2) *If $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$,*

$$\begin{cases} x = (-\alpha_2 - 2\alpha_3 - \alpha_5)t^{-1} + \cdots\, , \\ y = -t + \alpha_0 + \alpha_0(\alpha_0 - 2\alpha_3 + \alpha_4 - \alpha_5)t^{-1} + \cdots\, , \\ z = 1 - \alpha_4 t^{-1} + \cdots\, , \\ w = -\alpha_3 + \alpha_3(\alpha_3 - \alpha_4 + \alpha_5)t^{-1} + \cdots\, . \end{cases}$$

(3) *If* $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$,

$$\begin{cases} x = 1 - (\alpha_2 + 2\alpha_3 + \alpha_4)t^{-1} + \cdots , \\ y = -t - \alpha_0 + \alpha_0(\alpha_0 - 2\alpha_3 - \alpha_4 + \alpha_5)t^{-1} + \cdots , \\ z = -\alpha_5 t^{-1} + \cdots , \\ w = \alpha_3 + \alpha_3(\alpha_3 + \alpha_4 - \alpha_5)t^{-1} + \cdots . \end{cases}$$

(4) *If* $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$,

$$\begin{cases} x = 1 - (\alpha_2 + \alpha_4)t^{-1} + \cdots , \\ y = -t - \alpha_0 + \alpha_0(\alpha_0 + 2\alpha_3 - \alpha_4 + \alpha_5)t^{-1} + \cdots , \\ z = 1 - \alpha_4 t^{-1} + \cdots , \\ w = -\alpha_3 + \alpha_3(\alpha_3 - \alpha_4 + \alpha_5)t^{-1} + \cdots . \end{cases}$$

Proof. It can be proved by direct calculation. □

Proposition 1.8. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, *there exists a solution such that* y *has a pole at* $t = \infty$ *and* x, z, w *are all holomorphic at* $t = \infty$ *and* $b_{\infty,1} = -1$. *Moreover, assume that* $a_{\infty,0}$ *and* $c_{\infty,0}$ *are both determined. The solution is then unique.*

Proof. It can be proved in the same way as Proposition 1.2. □

Corollary 1.9. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, *there exists a solution such that* y *has a pole at* $t = \infty$ *and* x, z, w *are all holomorphic at* $t = \infty$ *and* $b_{\infty,1} = -1$. *Then,*

$$\begin{cases} y \equiv -t & \text{if } \alpha_0 = 0, \\ w \equiv 0 & \text{if } \alpha_3 = 0. \end{cases}$$

Let us deal with the case in which $b_{\infty,1} = -1/2$. From the proof of Proposition 1.6, it follows that $a_{\infty,0} = 1/2$. We first suppose that $c_{\infty,0} = 1/2$.

Proposition 1.10. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, *there exists a solution such that* y *has a pole of order one at* $t = \infty$ *and* x, z, w *are holomorphic at* $t = \infty$ *and* $c_{\infty,0} = 1/2$. *Then,* $a_{\infty,0} = 1/2,\ a_{\infty,-1} = -\alpha_0 + \alpha_1$ *and either of the following occurs:*
(1) $2\alpha_3 + \alpha_4 + \alpha_5 \neq 0$ *and* $b_{\infty,0} + d_{\infty,0} = -\alpha_4 + \alpha_5$ *and*

$$b_{\infty,0} = \frac{(\alpha_4 + \alpha_5)(-\alpha_4 + \alpha_5)}{2\alpha_3 + \alpha_4 + \alpha_5},\ d_{\infty,0} = \frac{2\alpha_3(-\alpha_4 + \alpha_5)}{2\alpha_3 + \alpha_4 + \alpha_5},$$
$$(1 - 2\alpha_3 - \alpha_4 - \alpha_5)c_{\infty,-1} = (\alpha_0 + \alpha_1)(-\alpha_0 + \alpha_1),$$

(2) $b_{\infty,0}+d_{\infty,0}=-\alpha_4+\alpha_5$ *and*

$$\begin{cases} \alpha_3=0, \ \alpha_4+\alpha_5=0, \\ or \\ -\alpha_4+\alpha_5=0, \ \alpha_3+\alpha_4=0. \end{cases}$$

Proof. By comparing the constant terms in

$$tx'=2x^2y+tx^2-2xy-\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}x \\ +(\alpha_2+\alpha_5)+2z\{(z-1)w+\alpha_3\},$$

we get

$$b_{\infty,0}+d_{\infty,0}=-\alpha_4+\alpha_5. \tag{1.5}$$

By comparing the coefficients of the term t in

$$ty'=-2xy^2+y^2-2txy+\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}y-\alpha_1 t,$$

we have $a_{\infty,-1}=-\alpha_0+\alpha_1$.

By comparing the constant terms in

$$tw'=-2zw^2+w^2-2tzw+\{t+(\alpha_5+\alpha_4)\}w-\alpha_3 t-2y(-w+2zw+\alpha_3),$$

we obtain

$$2\alpha_3 b_{\infty,0}-(\alpha_4+\alpha_5)d_{\infty,0}=0. \tag{1.6}$$

From (1.5) and (1.6), it follows that

$$(2\alpha_3+\alpha_4+\alpha_5)d_{\infty,0}=2\alpha_3(-\alpha_4+\alpha_5). \tag{1.7}$$

By comparing the coefficients of the term t^{-1} in

$$\begin{cases} tx'=2x^2y+tx^2-2xy-\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}x \\ \qquad\qquad +(\alpha_2+\alpha_5)+2z\{(z-1)w+\alpha_3\} \\ tz'=2z^2w+tz^2-2zw-\{t+(\alpha_5+\alpha_4)\}z+\alpha_5+2yz(z-1), \end{cases}$$

we obtain

$$\begin{cases} -2(\alpha_0+\alpha_1)a_{\infty,-1}+b_{\infty,-1}-4\alpha_3 c_{\infty,-1}+d_{\infty,-1}=0, \\ b_{\infty,-1}+2(-1+\alpha_4+\alpha_5)c_{\infty,-1}+d_{\infty,-1}=0. \end{cases}$$

We then find that

$$-2(\alpha_0+\alpha_1)(-\alpha_0+\alpha_1)+2(1-2\alpha_3-\alpha_4-\alpha_5)c_{\infty,-1}=0.$$

If $2\alpha_3+\alpha_4+\alpha_5 \neq 0$, it follows from (1.5) that

$$b_{\infty,0}=\frac{(\alpha_4+\alpha_5)(-\alpha_4+\alpha_5)}{2\alpha_3+\alpha_4+\alpha_5},\ d_{\infty,0}=\frac{2\alpha_3(-\alpha_4+\alpha_5)}{2\alpha_3+\alpha_4+\alpha_5},$$

and

$$(1-2\alpha_3-\alpha_4-\alpha_5)c_{\infty,-1}=(\alpha_0+\alpha_1)(-\alpha_0+\alpha_1).$$

If $2\alpha_3+\alpha_4+\alpha_5=0$, it follows from (1.7) that

$$\begin{cases}\alpha_3=0,\ \alpha_4+\alpha_5=0,\\ \text{or}\\ -\alpha_4+\alpha_5=0,\ \alpha_3+\alpha_4=0.\end{cases}$$

□

Proposition 1.11. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that y has a pole of order one at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$. Moreover, assume that $2\alpha_3+\alpha_4+\alpha_5 \notin \mathbb{Z}$. The solution is then unique.*

Proof. By Proposition 1.10, we may set

$$\begin{cases}x=1/2+a_{\infty,-1}t^{-1}+a_{\infty,-2}t^{-2}+\cdots,\\ y=-t/2+b_{\infty,0}+b_{\infty,-1}t^{-1}+\cdots,\\ z=1/2+c_{\infty,-1}t^{-1}+c_{\infty,-2}t^{-2}+\cdots,\\ w=d_{\infty,0}+d_{\infty,-1}t^{-1}+\cdots,\end{cases}$$

where $a_{\infty,-1}$, $b_{\infty,0}$, $c_{\infty,-1}$, and $d_{\infty,0}$ all have been determined.

We calculate the coefficients, $a_{\infty,-2}$, $b_{\infty,-1}$, $c_{\infty,-2}$, and $d_{\infty,-1}$. The other coefficients can be computed in the same way.

By comparing the constant terms in

$$ty'=-2xy^2+y^2-2txy+\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}\,y-\alpha_1 t,$$

we have $a_{\infty,-2}=-2b_{\infty,0}(2\alpha_2+2\alpha_3+\alpha_4+\alpha_5)$.

By comparing the coefficients of the term t^{-2} in

$$\begin{cases}tx'=2x^2y+tx^2-2xy-\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}\,x\\ \qquad\qquad+(\alpha_2+\alpha_5)+2z\,\{(z-1)w+\alpha_3\}\\ tz'=2z^2w+tz^2-2zw-\{t+(\alpha_5+\alpha_4)\}\,z+\alpha_5+2yz(z-1),\end{cases}$$

we obtain

$$\begin{cases} b_{\infty,-2} + d_{\infty,-2} = 4a_{\infty,-2} + 4a_{\infty,-1}^2 b_{\infty,0} \\ \qquad\qquad\qquad\quad -2(2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5)a_{\infty,-2} + 4c_{\infty,-1}^2 d_{\infty,0} + 4\alpha_3 c_{\infty,-2}, \\ b_{\infty,-2} + d_{\infty,-2} = 2(2 - \alpha_4 - \alpha_5)c_{\infty,-2} + 2c_{\infty,-1}^2(-\alpha_4 + \alpha_5), \end{cases}$$

which implies that

$$\begin{aligned} (2 - 2\alpha_3 - \alpha_4 - \alpha_5)c_{\infty,-2} &= 2a_{\infty,-2} + 2a_{\infty,-1}^2 b_{\infty,0} \\ &\quad - (2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5)a_{\infty,-2} + 2c_{\infty,-1}^2 d_{\infty,0} - c_{\infty,-1}^2(-\alpha_4 + \alpha_5). \end{aligned}$$

By comparing the coefficients of the term t^{-1} in

$$\begin{aligned} tz' &= 2z^2w + tz^2 - 2zw - \{t + (\alpha_5 + \alpha_4)\}\, z + \alpha_5 + 2yz(z-1), \\ tw' &= -2zw^2 + w^2 - 2tzw + \{t + (\alpha_5 + \alpha_4)\}\, w - \alpha_3 t - 2y(-w + 2zw + \alpha_3), \end{aligned}$$

we have

$$\begin{cases} b_{\infty,-1} + 2(-1 + \alpha_4 + \alpha_5)c_{\infty,-1} + d_{\infty,-1} = 0, \\ -2\alpha_3 b_{\infty,-1} + (\alpha_4 + \alpha_5 + 1)d_{\infty,-1} = 2c_{\infty,-1}d_{\infty,0}^2 + 4b_{\infty,0}c_{\infty,-1}d_{\infty,0}. \end{cases}$$

Since $2\alpha_3 + \alpha_4 + \alpha_5 \notin \mathbb{Z}$, $b_{\infty,-1}$ and $d_{\infty,-1}$ are both uniquely determined. $\square$

Proposition 1.12. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that y has a pole of order one at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} = 1/2$. Moreover, assume that $b_{\infty,0}, d_{\infty,0}$ are both determined and $2\alpha_3 + \alpha_4 + \alpha_5 = 0$. The solution is then unique.*

Proof. The proposition follows from the proof of Proposition 1.11. $\square$

Proposition 1.13. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that y has a pole of order one at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} = 1/2$. Moreover, assume that $c_{\infty,-1}$ is determined and $2\alpha_3 + \alpha_4 + \alpha_5 = 1$. The solution is then unique.*

Proof. The proposition follows from the proof of Proposition 1.11. $\square$

Corollary 1.14. *Suppose that $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ has a solution such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} = 1/2$. Moreover, assume that $2\alpha_3 + \alpha_4 + \alpha_5 = 0$, $d_{\infty,0} = 0$, and $-\alpha_0 + \alpha_1 = -\alpha_4 + \alpha_5 = 0$. Then,*

$$(x, y, z, w) = (1/2, -t/2, 1/2, 0).$$

Corollary 1.15. *Suppose that $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has a solution such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$. Moreover, assume that $\alpha_0=\alpha_1=1/2$, and $-\alpha_4+\alpha_5=2\alpha_3+\alpha_4+\alpha_5=0$. Then,*

$$(x,y,z,w)=(1/2,-t/2+b,1/2,d),$$

where b,d are both arbitrary complex numbers and satisfy $b+d=0$.

Let us treat the case where $b_{\infty,1}=-1/2$ and $c_{\infty,0}\neq 1/2$.

Proposition 1.16. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$. Then, $a_{\infty,0}=1/2,\ a_{\infty,-1}=-\alpha_0+\alpha_1, b_{\infty,1}=-1/2$ and one of the following occurs:*
(1) $\quad -\alpha_4+\alpha_5\neq 0$ *and*

$$b_{\infty,0}=\frac{(-\alpha_4+\alpha_5)(2\alpha_3+\alpha_4+\alpha_5)}{\alpha_4+\alpha_5},\ c_{\infty,0}=\frac{\alpha_5}{-\alpha_4+\alpha_5}\neq\frac{1}{2},\ d_{\infty,0}=\frac{-2\alpha_3(-\alpha_4+\alpha_5)}{\alpha_4+\alpha_5},$$

which implies that $\alpha_4+\alpha_5\neq 0$ and $b_{\infty,0}+d_{\infty,0}=-\alpha_4+\alpha_5\neq 0$,
(2) $\quad \alpha_4+\alpha_5\neq 0$ *and*

$$b_{\infty,0}=\frac{(\alpha_4+\alpha_5)(2\alpha_3+\alpha_4+\alpha_5)}{-\alpha_4+\alpha_5},\ c_{\infty,0}=\frac{\alpha_5}{\alpha_4+\alpha_5}\neq\frac{1}{2},\ d_{\infty,0}=-\frac{(\alpha_4+\alpha_5)(2\alpha_3+\alpha_4+\alpha_5)}{-\alpha_4+\alpha_5},$$

which implies that $-\alpha_4+\alpha_5\neq 0$ and $b_{\infty,0}+d_{\infty,0}=0$,
(3) $\quad \alpha_4=\alpha_5=0$ *and*

$$b_{\infty,0}=\frac{2\alpha_3}{2c_{\infty,0}-1},\ d_{\infty,0}=-\frac{2\alpha_3}{2c_{\infty,0}-1},$$

which implies that $b_{\infty,0}+d_{\infty,0}=0=-\alpha_4+\alpha_5$.

Proof. By comparing the coefficients of the term t in

$$ty'=-2xy^2+y^2-2txy+\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}\,y-\alpha_1 t,$$

we have

$$a_{\infty,-1}=-\alpha_0+\alpha_1.$$

By comparing the constant terms in

$$\begin{aligned}tx'=2x^2y+tx^2-2xy-\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}\,x\\ +(\alpha_2+\alpha_5)+2z\,\{(z-1)w+\alpha_3\}\,,\end{aligned}$$

we obtain

$$b_{\infty,0} - 4c_{\infty,0}(c_{\infty,0} - 1)d_{\infty,0} = -2\alpha_3 - \alpha_4 + \alpha_5 + 4\alpha_3 c_{\infty,0}. \tag{1.8}$$

By comparing the constant terms in

$$tz' = 2z^2w + tz^2 - 2zw - \{t + (\alpha_5 + \alpha_4)\}\, z + \alpha_5 + 2yz(z-1),$$

we have

$$2c_{\infty,0}(c_{\infty,0} - 1)d_{\infty,0} - (\alpha_5 + \alpha_4)c_{\infty,0} + \alpha_5 + 2b_{\infty,0}c_{\infty,0}(c_{\infty,0} - 1) = 0. \tag{1.9}$$

From (1.8) and (1.9), it follows that

$$b_{\infty,0} = \frac{1}{2c_{\infty,0} - 1}(2\alpha_3 + \alpha_4 + \alpha_5). \tag{1.10}$$

By comparing the constant terms in

$$tw' = -2zw^2 + w^2 - 2tzw + \{t + (\alpha_5 + \alpha_4)\}\, w - \alpha_3 t - 2y(-w + 2zw + \alpha_3),$$

we have

$$(2c_{\infty,0} - 1)^2 d_{\infty,0}^2 + (2c_{\infty,0} - 1)(4\alpha_3 + \alpha_4 + \alpha_5)d_{\infty,0} + 2\alpha_3(2\alpha_3 + \alpha_4 + \alpha_5) = 0,$$

which implies that

$$d_{\infty,0} = \frac{-2\alpha_3}{2c_{\infty,0} - 1},\ \frac{-2\alpha_3 - \alpha_4 - \alpha_5}{2c_{\infty,0} - 1}. \tag{1.11}$$

If $d_{\infty,0} = -2\alpha_3/(2c_{\infty,0} - 1)$, it follows from (1.9) and (1.10) that

$$(-\alpha_4 + \alpha_5)c_{\infty,0} = \alpha_5. \tag{1.12}$$

If $-\alpha_4 + \alpha_5 \neq 0$, we have $c_{\infty,0} = \alpha_5/(-\alpha_4 + \alpha_5) \neq 1/2$, which implies that $\alpha_4 + \alpha_5 \neq 0$ and

$$b_{\infty,0} = \frac{(-\alpha_4 + \alpha_5)(2\alpha_3 + \alpha_4 + \alpha_5)}{\alpha_4 + \alpha_5},\ d_{\infty,0} = \frac{-2\alpha_3(-\alpha_4 + \alpha_5)}{\alpha_4 + \alpha_5}.$$

If $-\alpha_4 + \alpha_5 = 0$, it follows from (1.10), (1.11), (1.12) that $\alpha_4 = \alpha_5 = 0$ and

$$b_{\infty,0} = \frac{2\alpha_3}{2c_{\infty,0} - 1},\ d_{\infty,0} = \frac{-2\alpha_3}{2c_{\infty,0} - 1}.$$

If $d_{\infty,0} = (-2\alpha_3 - \alpha_4 - \alpha_5)/(2c_{\infty,0} - 1)$, it follows from (1.9) and (1.10) that

$$(\alpha_4 + \alpha_5)c_{\infty,0} = \alpha_5. \tag{1.13}$$

If $\alpha_4+\alpha_5 \neq 0$, we obtain $c_{\infty,0}=\alpha_5/(\alpha_4+\alpha_5) \neq 1/2$, which implies that $-\alpha_4+\alpha_5 \neq 0$ and

$$b_{\infty,0}=\frac{(\alpha_4+\alpha_5)(2\alpha_3+\alpha_4+\alpha_5)}{-\alpha_4+\alpha_5},\ d_{\infty,0}=-\frac{(\alpha_4+\alpha_5)(2\alpha_3+\alpha_4+\alpha_5)}{-\alpha_4+\alpha_5},$$

If $\alpha_4+\alpha_5=0$, it follows from (1.10), (1.11), (1.13) that $\alpha_4=\alpha_5=0$ and

$$b_{\infty,0}=\frac{2\alpha_3}{2c_{\infty,0}-1},\ d_{\infty,0}=\frac{-2\alpha_3}{2c_{\infty,0}-1}.$$

□

Proposition 1.17. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$. Moreover, assume that $\alpha_4=\alpha_5=0$ and $c_{\infty,0}$ is determined. The solution is then unique.*

Proof. By Proposition 1.16, we set

$$\begin{cases} x=1/2+a_{\infty,-1}t^{-1}+a_{\infty,-2}t^{-2}+\cdots, \\ y=-t/2+b_{\infty,0}+b_{\infty,-1}t^{-1}+\cdots, \\ z=c_{\infty,0}+c_{\infty,-1}t^{-1}+\cdots, \\ w=d_{\infty,0}+d_{\infty,-1}t^{-1}+\cdots, \end{cases}$$

where $a_{\infty,0}$, $b_{\infty,0}$, $c_{\infty,0}$ and $d_{\infty,0}$ have all been determined. We compute the coefficients $a_{\infty,-2}$, $b_{\infty,-1}$, $c_{\infty,-1}$ and $d_{\infty,-1}$. The other coefficients can be calculated in the same way.

By comparing the constant terms in

$$ty'=-2xy^2+y^2-2txy+\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}y-\alpha_1 t,$$

we have $a_{\infty,-2}=-2b_{\infty,0}(2\alpha_2+2\alpha_3+\alpha_4+\alpha_5)$.

By comparing the coefficients of the term t^{-1} in

$$\begin{aligned} tx'=2x^2y+tx^2-2xy-\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}x \\ +(\alpha_2+\alpha_5)+2z\{(z-1)w+\alpha_3\}, \end{aligned}$$

we obtain

$$\begin{aligned} -\frac{1}{2}b_{\infty,-1}+(\alpha_0+\alpha_1)a_{\infty,-1}+2c_{\infty,-1}d_{\infty,0}(2c_{\infty,0}-1) \\ +2c_{\infty,0}(c_{\infty,0}-1)d_{\infty,-1}+2\alpha_3 c_{\infty,-1}=0. \end{aligned} \tag{1.14}$$

By comparing the coefficients of the term t^{-1} in

$$tz'=2z^2w+tz^2-2zw-\{t+(\alpha_5+\alpha_4)\}z+\alpha_5+2yz(z-1),$$

we have

$$2c_{\infty,-1}d_{\infty,0}(2c_{\infty,0}-1)+2c_{\infty,0}(c_{\infty,0}-1)d_{\infty,-1}+2b_{\infty,0}c_{\infty,-1}(2c_{\infty,0}-1)$$
$$+2b_{\infty,-1}c_{\infty,0}(c_{\infty,0}-1)-(\alpha_5+\alpha_4)c_{\infty,-1}=-c_{\infty,-1}. \tag{1.15}$$

From equations (1.14) and (1.15), it follows that

$$b_{\infty,-1}=\frac{2}{(2c_{\infty,0}-1)^2}\left[(\alpha_0+\alpha_1)a_{\infty,-1}-(2\alpha_3+\alpha_4+\alpha_5+1)c_{\infty,-1}\right]. \tag{1.16}$$

By comparing the coefficients of the term t^{-1} in

$$tw'=-2zw^2+w^2-2tzw+\{t+(\alpha_5+\alpha_4)\}\,w-\alpha_3t-2y(-w+2zw+\alpha_3),$$

we have

$$-2\alpha_3b_{\infty,-1}-d_{\infty,-1}=-2c_{\infty,-1}d_{\infty,0}^2, \tag{1.17}$$

where we have used the condition, $\alpha_4=\alpha_5=0$. Therefore, $d_{\infty,-1}$ is given by

$$d_{\infty,-1}=\frac{4\alpha_3}{(2c_{\infty,0}-1)^2}\left[-(\alpha_0+\alpha_1)a_{\infty,-1}+c_{\infty,-1}\right]. \tag{1.18}$$

Thus, from equation 1.15, we obtain

$$c_{\infty,-1}=4c_{\infty,0}(c_{\infty,0}-1)(\alpha_0+\alpha_1)a_{\infty,-1}(2\alpha_3-1),$$

which determines $b_{\infty,-1}$ and $d_{\infty,-1}$. □

Corollary 1.18. *Suppose that for $D_5^{(1)}(1/2,1/2,\alpha_2,\alpha_3,0,0)$, there exists a solution such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$. Then,*

$$x=\frac{1}{2},\ y=-\frac{t}{2}+\frac{2\alpha_3}{2c-1},\ z=c,\ w=-\frac{2\alpha_3}{2c-1},\ c\neq\frac{1}{2}$$

and the solution is unique.

Remark

If $\alpha_3\neq 0$, s_3 transforms the parameters and solution of Corollary 1.18 into those of Corollary 1.15. If $\alpha_3=0$, $s_3s_2s_1s_0$ does so.

Corollary 1.19. *Suppose that for $D_5^{(1)}(1/2-\alpha_2,1/2-\alpha_2,\alpha_2,0,0,0)$, there exists a solution such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$. Then,*

$$(x,y,z,w)=(1/2,-t/2,c,0),\ c\neq 1/2,$$

and the solution is unique.

Remark

If $\alpha_2 \neq 0$, $s_3 s_2$ transforms the parameters and solution of Corollary 1.19 into those of Corollary 1.15. If $\alpha_2 = 0$, $s_3 s_2 s_1 s_0$ does so.

1.2.3 The case where z has a pole at $t = \infty$

Proposition 1.20. *For $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists no solution such that z has a pole at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$.*

Proof. It can be easily checked. □

1.2.4 The case where w has a pole at $t = \infty$

Lemma 1.21. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$. w then has a pole of order one at $t = \infty$.*

Proof. It can be easily checked. □

Lemma 1.22. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that w has a pole of order one at $t = \infty$ and x, y, z are holomorphic at $t = \infty$. Then, $d_{\infty,1} = -1, -1/2$.*

Proof. It can be proved by direct calculation. □

Let us first suppose that $d_{\infty,1} = -1$.

Proposition 1.23. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$ and $d_{\infty,1} = -1$. Then, $a_{\infty,0} = 0, 1$ and $c_{\infty,0} = 0, 1$.*
(1) *If $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$,*

$$\begin{cases} x = (\alpha_2 - \alpha_5)t^{-1} + \cdots, \\ y = \alpha_1 + \alpha_1(-\alpha_1 - 2\alpha_3 - \alpha_4 - 3\alpha_5)t^{-1} + \cdots, \\ z = -\alpha_5 t^{-1} + \cdots, \\ w = -t + (\alpha_0 - \alpha_1 + 2\alpha_2 + \alpha_3) \\ \qquad + \{(\alpha_0 - \alpha_1 + 2\alpha_2 + \alpha_3)(\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 - \alpha_5) + 2\alpha_1(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)\} t^{-1} + \cdots. \end{cases}$$

(2) *If* $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$,

$$\begin{cases} x = 1 + (\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5)t^{-1} + \cdots, \\ y = -\alpha_1 - \alpha_1(\alpha_1 - 2\alpha_3 - \alpha_4 - 3\alpha_5)t^{-1} + \cdots, \\ z = -\alpha_5 t^{-1} + \cdots, \\ w = -t + (\alpha_0 + 3\alpha_1 + 2\alpha_2 + \alpha_5) \\ \qquad + \{(\alpha_0 + 3\alpha_1 + 2\alpha_2 + \alpha_3)(\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 - \alpha_5) + 2\alpha_1(\alpha_1 - \alpha_3 - \alpha_4 - \alpha_5)\}\, t^{-1} + \cdots. \end{cases}$$

(3) *If* $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$,

$$\begin{cases} x = (\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)t^{-1} + \cdots, \\ y = \alpha_1 + \alpha_1(-\alpha_1 + 2\alpha_3 + 3\alpha_4 + \alpha_5)t^{-1} + \cdots, \\ z = 1 - \alpha_4 t^{-1} + \cdots, \\ w = -t - (\alpha_0 + 3\alpha_1 + 2\alpha_2 + \alpha_3) \\ \qquad + \{(\alpha_0 + 3\alpha_1 + 2\alpha_2 + \alpha_3)(\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 - \alpha_4 + \alpha_5) + 2\alpha_1(\alpha_1 - \alpha_3 - \alpha_4 - \alpha_5)\}\, t^{-1} + \cdots. \end{cases}$$

(4) *If* $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$,

$$\begin{cases} x = 1 + (\alpha_2 - \alpha_4)t^{-1} + \cdots, \\ y = -\alpha_1 - \alpha_1(\alpha_1 + 2\alpha_3 + 3\alpha_4 + \alpha_5)t^{-1} + \cdots, \\ z = 1 - \alpha_4 t^{-1} + \cdots, \\ w = -t + (-\alpha_0 + \alpha_1 - 2\alpha_2 - \alpha_3) \\ \quad + \{(-\alpha_0 + \alpha_1 - 2\alpha_2 - \alpha_3)(-\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 + \alpha_4 - \alpha_5) + 2\alpha_1(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)\}\, t^{-1} + \cdots. \end{cases}$$

Proof. It can be proved by direct calculations. □

Proposition 1.24. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$ and $d_{\infty,1} = -1$. Moreover, assume that $a_{\infty,0}$ and $c_{\infty,0}$ are both determined. It is then unique.*

Proof. It can be proved in the same way as Proposition 1.2. □

Corollary 1.25. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$ and $d_{\infty,1} = -1$. Then,*

$$y \equiv 0 \ \text{if} \ \alpha_1 = 0.$$

Now, let us suppose that $d_{\infty,1} = -1/2$.

Proposition 1.26. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that w has a pole at $t=\infty$ and x,y,z are all holomorphic at $t=\infty$ and $d_{\infty,1}=-1/2$. Then, $\alpha_1=0$ and*

$$\begin{cases} x=\dfrac{1}{2}+a_{\infty,-1}t^{-1}+\cdots\,, \\ y=b_{\infty,0}+b_{\infty,-1}t^{-1}+\cdots\,, \\ z=\dfrac{1}{2}+(-\alpha_0-\alpha_1-2\alpha_2)t^{-1}+\cdots\,, \\ w=-\dfrac{1}{2}t+d_{\infty,0}+d_{\infty,-1}t^{-1}+\cdots\,, \end{cases}$$

where

$$b_{\infty,0}+d_{\infty,0}=-\alpha_4+\alpha_5.$$

Proof. It can be proved by direct calculations. □

1.3 The case where two of (x,y,z,w) have a pole at $t=\infty$

In this subsection, we deal with the case in which two of (x,y,z,w) have a pole at $t=\infty$. For this purpose, we consider the following six cases:
(1) x,y both have a pole at $t=\infty$ and z,w are both holomorphic at $t=\infty$,
(2) x,z have a pole at $t=\infty$ and y,w are both holomorphic at $t=\infty$,
(3) x,w have a pole at $t=\infty$ and y,z are both holomorphic at $t=\infty$,
(4) y,z have a pole at $t=\infty$ and x,w are both holomorphic at $t=\infty$,
(5) y,w have a pole at $t=\infty$ and x,z are both holomorphic at $t=\infty$,
(6) z,w have a pole at $t=\infty$ and x,y are both holomorphic at $t=\infty$.
If case (4) or (5) occurs, we can obtain necessary conditions for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ to have a meromorphic solution at $t=\infty$.

1.3.1 The case where x,y have a pole at $t=\infty$

Proposition 1.27. *For $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists no solution such that x,y both have a pole at $t=\infty$ and z,w are both holomorphic at $t=\infty$.*

Proof. It can be easily checked. □

1.3.2 The case where x,z have a pole at $t=\infty$

Proposition 1.28. *For $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists no solution such that x,z both have a pole at $t=\infty$ and y,w are both holomorphic at $t=\infty$.*

Proof. It can be easily checked. □

1.3.3 The case where x, w have a pole at $t = \infty$

Proposition 1.29. *For $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists no solution such that x, w both have a pole at $t = \infty$ and y, z are both holomorphic at $t = \infty$.*

Proof. It can be easily checked. □

1.3.4 The case where y, z have a pole at $t = \infty$

Lemma 1.30. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that y, z both have a pole at $t = \infty$ and x, w are both holomorphic at $t = \infty$. y then has a pole of order one at $t = \infty$.*

Proof. It can be proved by direct calculation. □

Proposition 1.31. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that y, z both have a pole at $t = \infty$ and x, w are both holomorphic at $t = \infty$. Then,*

$$\begin{cases} x = \dfrac{1}{2} + (-\alpha_0 + \alpha_1)t^{-1} + \cdots, \\ y = -\dfrac{1}{2}t + \dfrac{1}{2c_{\infty,n}}(n + 2\alpha_3 + \alpha_4 + \alpha_5)t^{-n} + \cdots, \\ z = c_{\infty,n}t^n + \cdots, \\ w = -\dfrac{\alpha_3}{c_{\infty,n}}t^{-n} + \cdots, \end{cases}$$

where $c_{\infty,n}$ is not zero.

Proof. It can be proved by direct calculation. □

Corollary 1.32. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that y, z both have a pole at $t = \infty$ and x, w are both holomorphic at $t = \infty$. Moreover, assume that $\alpha_3 \neq 0$. s_3 then transforms (x, y, z, w) into a solution such that $s_3(y)$ has a pole of order one at $t = \infty$ and all of $s_3(x, z, w)$ are holomorphic at $t = \infty$.*

Proof. By direct calculation, we find that $s_3(x, y, z, w)$ is a solution of $D_5^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5 + \alpha_3)$ and one of the following occurs:
(1) $s_3(y)$ has a pole of order one at $t = \infty$ and all of $s_3(x, z, w)$ are holomorphic at $t = \infty$,
(2) $s_3(y)$ has a pole of order one at $t = \infty$ and $s_3(z)$ has a pole of order m $(1 \leq m < n)$ and both of $s_3(x, w)$ are holomorphic at $t = \infty$.
If case (1) occurs, the corollary is proved.

Let us suppose that case (2) occurs. It then follows from Proposition 1.31 that

$$s_3(z) = c'_{\infty,m}t^m + \cdots, \ \ s_3(w) = -\frac{(-\alpha_3)}{c'_{\infty,m}}t^{-m} + \cdots .$$

On the other hand, by direct calculation, we find that

$$s_3(w) = -\frac{\alpha_3}{c_{\infty,n}}t^{-n} + \cdots ,$$

which is contradiction. □

By Corollary 1.32, we can obtain necessary conditions.

Proposition 1.33. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that y, z both have a pole at $t=\infty$ and x, w are both holomorphic at $t=\infty$. Either of the following then occurs:* (1) $\quad -\alpha_4+\alpha_5=0$, $\quad$ (2) $\quad \alpha_4+\alpha_5=0$.

Proof. Let us first suppose that $\alpha_3 \neq 0$. We prove that if $\alpha_3 \neq 0$, $-\alpha_4+\alpha_5=0$ or $\alpha_4+\alpha_5=0$. From Corollary 1.32, it follows that for $D_5^{(1)}(\alpha_0, \alpha_1, \alpha_2+\alpha_3, -\alpha_3, \alpha_4+\alpha_3, \alpha_5+\alpha_3)$, $s_3(y)$ has a pole of order one at $t=\infty$ and all of $s_3(x, z, w)$ are holomorphic at $t=\infty$.

Assume that for $s_3(x, y, z, w)$, $c_{\infty,0}=1/2$. Then, case (1) or (2) occurs in Proposition 1.10. If case (1) occurs, it follows that $-\alpha_4+\alpha_5=0$, because for $s_3(x, y, z, w)$, $d_{\infty,0}=0$. If case (2) occurs, then $\alpha_3=0$, which is contradiction.

Assume that for $s_3(x, y, z, w)$, $c_{\infty,0} \neq 1/2$. Then, one of cases (1), (2), (3) occurs in Proposition 1.16. If case (1) occurs, then $\alpha_3=0$, because for $s_3(x, y, z, w)$, $d_{\infty,0}=0$. However, this is impossible. If case (2) occurs, then $\alpha_4+\alpha_5=0$, because for $s_3(x, y, z, w)$, $d_{\infty,0}=0$. If case (3) occurs, then $\alpha_3+\alpha_4=\alpha_3+\alpha_5=0$, which implies that $-\alpha_4+\alpha_5=0$.

Let us suppose that $\alpha_3=0$. Now, we can assume that $\alpha_4 \neq 0$ or $\alpha_5 \neq 0$. If $\alpha_4 \neq 0$, $s_4(x, y, z, w)$ is then a solution of $D_5^{(1)}(\alpha_0, \alpha_1, \alpha_2, \alpha_4, -\alpha_4, \alpha_5)$ such that $s_4(y)$ has a pole of order one at $t=\infty$ and $s_4(z)$ has a pole of order n and both of $s_4(x, w)$ are holomorphic at $t=\infty$. Therefore, it follows that $\alpha_4+\alpha_5=0$ or $-\alpha_4+\alpha_5=0$. If $\alpha_5 \neq 0$, we use s_5 in the same way and can obtain the necessary conditions. □

1.3.5 The case in which y, w have a pole at $t=\infty$

Lemma 1.34. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that y, w both have a pole at $t=\infty$ and x, z are both holomorphic at $t=\infty$. Both of y, w then have a pole of order n $(n \geq 1)$ at $t=\infty$.*

Proof. It can be easily checked. □

Let us treat the case in which y, w have a pole of order $n \geq 2$ at $t=\infty$.

Proposition 1.35. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$ there exists a solution such that y, w both have a pole of order n $(n \geq 2)$ at $t = \infty$ and x, z are both holomorphic at $t = \infty$. Then,*

$$\begin{cases} x = \dfrac{1}{2} + \dfrac{1}{2b_{\infty,n}}(-n + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5)t^{-n} + \cdots , \\ y = b_{\infty,n}t^n + \cdots + b_{\infty,2}t^2 + b_{\infty,1}t + b_{\infty,0} + \cdots , \\ z = \dfrac{1}{2} + \dfrac{1}{2b_{\infty,n}}(-n + 2\alpha_3 + \alpha_4 + \alpha_5)t^{-n} + \cdots , \\ w = d_{\infty,n}t^n + \cdots + d_{\infty,2}t^2 + d_{\infty,1}t + d_{\infty,0} + \cdots , \end{cases}$$

where

$$b_{\infty,k} + d_{\infty,k} = 0 \ (2 \leq k \leq n), \ \ b_{\infty,1} + d_{\infty,1} = -\frac{1}{2} \ \textit{and} \ b_{\infty,0} + d_{\infty,0} = -\alpha_4 + \alpha_5.$$

Proof. It can be proved by direct calculation. □

Corollary 1.36. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$ there exists a solution such that y, w both have a pole of order n $(n \geq 2)$ at $t = \infty$ and x, z are both holomorphic at $t = \infty$. Moreover, assume that $\alpha_2 \neq 0$. s_2 then transforms (x, y, z, w) into a solution such that one of the following occurs:*
(1) $s_2(y)$ has a pole of order one at $t = \infty$ and all of $s_2(x, z, w)$ are holomorphic at $t = \infty$;
(2) $s_2(w)$ has a pole of order one at $t = \infty$ and all of $s_2(x, y, z)$ are holomorphic at $t = \infty$;
(3) both of $s_2(y, w)$ have a pole of order one at $t = \infty$ and both of $s_2(x, z)$ are holomorphic at $t = \infty$.

Proof. If $n = 2$, by direct calculation, we can prove the corollary. Suppose that $n \geq 3$ and none of cases (1), (2) and (3) occurs. $s_2(x, y, z, w)$ is then a solution of $D_5^{(1)}(\alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5)$ such that both of $s_2(y, w)$ have a pole of order m $(2 \leq m < n)$ at $t = \infty$ and both of $s_2(x, z)$ are holomorphic at $t = \infty$. Moreover, by direct calculation, we find that

$$s_2(y) = b'_{\infty,m}t^m + \cdots , \ s_2(x) - s_2(z) = \frac{\alpha_2}{b_{\infty,n}}t^{-n} + \cdots .$$

On the other hand, it follows from Proposition 1.35 that

$$s_2(x) - s_2(z) = \frac{-\alpha_2}{b'_{\infty,m}}t^{-m} + \cdots ,$$

which is contradiction. □

Now, let us suppose that y, w have a pole of order one at $t = \infty$.

Proposition 1.37. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that y,w both have a pole of order one at $t=\infty$ and x,z are both holomorphic at $t=\infty$. One of the following then occurs:*
(1) $(b_{\infty,1},d_{\infty,1})=(-1,1)$,
(2) $b_{\infty,1}+d_{\infty,1}=-1/2$ *and* $a_{\infty,0}=c_{\infty,0}=1/2$.

Proof. It can be proved by direct calculation. □

We first deal with the case where $(b_{\infty,1},d_{\infty,1})=(-1,1)$.

Proposition 1.38. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that y,w both have a pole of order one at $t=\infty$ and x,z are both holomorphic at $t=\infty$ and $b_{\infty,1}=-1,\ d_{\infty,1}=1$. Then, $a_{\infty,0}=0,1$ and $c_{\infty,0}=0,1$.*
(1) *If* $(a_{\infty,0},c_{\infty,0})=(0,0)$,

$$\begin{cases} x=(-\alpha_2+\alpha_5)t^{-1}+\cdots, \\ y=-t+\alpha_0+\alpha_0(\alpha_0+2\alpha_3+\alpha_4+3\alpha_5)t^{-1}+\cdots, \\ z=\alpha_5 t^{-1}+\cdots, \\ w=t+(-\alpha_0+\alpha_1+2\alpha_2+\alpha_3)+\cdots. \end{cases}$$

(2) *If* $(a_{\infty,0},c_{\infty,0})=(0,1)$,

$$\begin{cases} x=(-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5)t^{-1}+\cdots, \\ y=-t+\alpha_0+\alpha_0(\alpha_0-2\alpha_3-3\alpha_4-\alpha_5)t^{-1}+\cdots, \\ z=1+\alpha_4 t^{-1}+\cdots, \\ w=t+(-3\alpha_0-\alpha_1-2\alpha_2-\alpha_3)+\cdots. \end{cases}$$

(3) *If* $(a_{\infty,0},c_{\infty,0})=(1,0)$,

$$\begin{cases} x=1+(-\alpha_2-2\alpha_3-\alpha_4-2\alpha_5)t^{-1}+\cdots, \\ y=-t-\alpha_0+\alpha_0(\alpha_0-2\alpha_3-\alpha_4-3\alpha_5)t^{-1}+\cdots, \\ z=\alpha_5 t^{-1}+\cdots, \\ w=t+(3\alpha_0+\alpha_1+2\alpha_2+\alpha_3)+\cdots. \end{cases}$$

(4) *If* $(a_{\infty,0},c_{\infty,0})=(1,1)$,

$$\begin{cases} x=1+(-\alpha_2+\alpha_4)t^{-1}+\cdots, \\ y=-t-\alpha_0+\alpha_0(\alpha_0+2\alpha_3+3\alpha_4+\alpha_5)t^{-1}+\cdots, \\ z=1+\alpha_4 t^{-1}+\cdots, \\ w=t+(\alpha_0-\alpha_1-2\alpha_2-\alpha_3)+\cdots. \end{cases}$$

Proof. It can be proved by direct calculation. $\square$

Proposition 1.39. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that y, w both have a pole of order one at $t = \infty$ and x, z are both holomorphic at $t = \infty$ and $b_{\infty,1} = -1,\ d_{\infty,1} = 1$. Moreover, assume that $a_{\infty,0}$ and $c_{\infty,0}$ are both determined. The solution is then unique.*

Proof. It can be proved in the same way as Proposition 1.2. $\square$

Corollary 1.40. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that y, w both have a pole of order one at $t = \infty$ and x, z are both holomorphic at $t = \infty$ and $b_{\infty,1} = -1,\ d_{\infty,1} = 1$. Then,*

$$y \equiv -t \ \textit{if}\ \ \alpha_0 = 0.$$

Let us treat the case where $b_{\infty,1} + d_{\infty,1} = -1/2$ and $a_{\infty,0} = c_{\infty,0} = 1/2$.

Proposition 1.41. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that y, w both have a pole of order one at $t = \infty$ and x, z are both holomorphic at $t = \infty$ and $b_{\infty,1} + d_{\infty,1} = -1/2$ and $a_{\infty,0} = c_{\infty,0} = 1/2$. Then, $b_{\infty,0} + d_{\infty,0} = -\alpha_4 + \alpha_5$ and one of the following occurs:*
(1) $\alpha_0 - \alpha_1 \neq 0$ *and*

$$a_{\infty,-1} = -\alpha_0+\alpha_1,\ b_{\infty,1} = \frac{-\alpha_1}{-\alpha_0+\alpha_1},\ c_{\infty,-1} = \frac{(-\alpha_0+\alpha_1)(\alpha_0+\alpha_1+2\alpha_2)}{\alpha_0+\alpha_1},\ d_{\infty,1} = \frac{\alpha_0+\alpha_1}{2(-\alpha_0+\alpha_1)},$$

(2) $\alpha_0 + \alpha_1 \neq 0$ *and*

$$a_{\infty,-1} = 0,\ b_{\infty,1} = \frac{-\alpha_1}{\alpha_0+\alpha_1},\ c_{\infty,-1} = \frac{(\alpha_0+\alpha_1)(\alpha_0+\alpha_1+2\alpha_2)}{-\alpha_0+\alpha_1},\ d_{\infty,1} = \frac{-\alpha_0+\alpha_1}{2(\alpha_0+\alpha_1)},$$

(3) $\alpha_0 = \alpha_1 = 0$ *and*

$$a_{\infty,-1} = 0,\ c_{\infty,-1} = \frac{-2\alpha_2}{2b_{\infty,1}+1}.$$

Proof. $\pi_2(x, y, z, w)$ is a solution of $D_5^{(1)}(\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0)$ such that $\pi_2(y)$ has a pole of order one at $t = \infty$ and $\pi_2(x), \pi_2(z), \pi_2(w)$ are holomorphic at $t = \infty$. Moreover, for $\pi_2(x, y, z, w)$, $b_{\infty,1} = -1/2$ and $c_{\infty,0} \neq 1/2$. Therefore, the proposition follows from Proposition 1.16. $\square$

Proposition 1.42. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that y, w both have a pole of order n $(n \ge 2)$ at $t = \infty$ and x, z are both holomorphic at $t = \infty$. One of the following then occurs:* (1) $-\alpha_0 + \alpha_1 = 0$, (2) $\alpha_0 + \alpha_1 = 0$.

Proof. We first prove that if $\alpha_2 \neq 0$, then $-\alpha_0 + \alpha_1 = 0$ or $\alpha_0 + \alpha_1 = 0$. Since $\alpha_2 \neq 0$, $s_2(x, y, z, w)$ is a solution of $D_5^{(1)}(\alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5)$ such that one of the following occurs:
(i) $s_2(y)$ has a pole of order one at $t = \infty$ and all of $s_2(x, z, w)$ are holomorphic at $t = \infty$,
(ii) $s_2(w)$ has a pole of order one at $t = \infty$ and all of $s_2(x, y, z)$ are holomorphic at $t = \infty$,
(iii) $s_2(y, w)$ have a pole of order one at $t = \infty$ and both of $s_2(x, z)$ are holomorphic at $t = \infty$.

If case (i) occurs, then $-\alpha_0 + \alpha_1 = 0$, because for $s_2(x, y, z, w)$, $a_{\infty,-1} = 0$. If case (ii) occurs, then $\alpha_0 + \alpha_1 = 0$, because for $s_2(x, y, z, w)$, $c_{\infty,-1} = 0$. If case (iii) occurs, then $\alpha_0 + \alpha_1 = 0$, because for $s_2(x, y, z, w)$, $c_{\infty,-1} = 0$.

Let us suppose that $\alpha_2 = 0$. We can then assume that $\alpha_0 \neq 0$, or $\alpha_1 \neq 0$. If $\alpha_0 \neq 0$, $s_0(x, y, z, w)$ is then a solution of $D_5^{(1)}(-\alpha_0, \alpha_1, \alpha_0, \alpha_3, \alpha_4, \alpha_5)$ such that both of $s_0(y, w)$ have a pole of order n at $t = \infty$ and both of $s_0(x, z)$ are holomorphic at $t = \infty$. It then follows from the above discussions that $\alpha_0 + \alpha_1 = 0$ or $-\alpha_0 + \alpha_1 = 0$.

If $\alpha_1 \neq 0$, we use s_1 in the same way and can obtain the necessary conditions. □

1.4 The case where three of (x, y, z, w) have a pole at $t = \infty$

In this subsection, we treat the case in which three of (x, y, z, w) have a pole at $t = \infty$. For this purpose, we consider the following four cases:
(1) y, z, w all have a pole at $t = \infty$ and x is holomorphic at $t = \infty$,
(2) x, z, w all have a pole at $t = \infty$ and y is holomorphic at $t = \infty$,
(3) x, y, w all have a pole at $t = \infty$ and z is holomorphic at $t = \infty$,
(4) x, y, z all have a pole at $t = \infty$ and w is holomorphic at $t = \infty$.

1.4.1 The case where y, z, w have a pole at $t = \infty$

Proposition 1.43. *For $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists no solution such that y, z, w all have a pole at $t = \infty$ and x is holomorphic at $t = \infty$.*

Proof. It can be easily checked. □

1.4.2 The case where x, z, w have a pole at $t = \infty$

Proposition 1.44. *For $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists no solution such that x, z, w all have a pole at $t = \infty$ and y is holomorphic at $t = \infty$.*

Proof. It can be easily checked. □

1.4.3 The case where x, y, w have a pole at $t = \infty$

Proposition 1.45. *For $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists no solution such that x, y, w all have a pole at $t = \infty$ and z is holomorphic at $t = \infty$.*

Proof. It can be easily checked. □

1.4.4 The case where x, y, z have a pole at $t = \infty$

Proposition 1.46. *For $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists no solution such that x, y, z all have a pole at $t = \infty$ and w is holomorphic at $t = \infty$.*

Proof. It can be easily checked. □

1.5 The case where all of (x, y, z, w) have a pole at $t = \infty$

In this subsection, we treat the case in which all of (x, y, z, w) have a pole at $t = \infty$.

Proposition 1.47. *For $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists no solution such that x, y, z, w have a pole at $t = \infty$.*

Proof. It can be easily checked. □

1.6 Summary

1.6.1 Summary for type A

If $b_{\infty,1} + d_{\infty,1} \neq -1/2$, we say that (x, y, z, w) is a solution of type A.

Proposition 1.48. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a meromorphic solution at $t = \infty$ of type A. One of the following then occurs:*
(1) *x, y, z, w are all holomorphic at $t = \infty$,*
(2) *y has a pole of order one at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$,*
(3) *w has a pole of order one at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$,*
(4) *y, w both have a pole of order one at $y = \infty$ and x, z are both holomorphic at $t = \infty$.*

If $a_{\infty,0}$ and $c_{\infty,0}$ are both determined in each of cases (1), (2), (3) *and* (4)*, the solution is unique.*

1.6.2 Summary for type B

If $b_{\infty,1} + d_{\infty,1} = -1/2$, we say that (x, y, z, w) is a solution of type B.

Proposition 1.49. (1) *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a meromorphic solution at $t = \infty$ of type B. One of the following then occurs:*
(i) *y has a pole of order one at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$,*
(ii) *w has a pole of order one at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$,*
(iii) *y has a pole of order one at $t = \infty$ and z has a pole of order n $(n \ge 1)$ at $t = \infty$ and x, w are both holomorphic at $t = \infty$,*
(iv) *y, w both have a pole of order n $(n \ge 1)$ at $t = \infty$ and x, z are both holomorphic at $t = \infty$.*
(2) *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution of type B. $y + w$ then has a pole of order one at $t = \infty$ and $b_{\infty,0} + d_{\infty,0} = 0, -\alpha_4 + \alpha_5$.*

In order to classify the rational solutions of type B, we find it important to investigate the case where y has a pole at $t = \infty$ and $c_{\infty,0} = 1/2$.

2 Meromorphic solutions at $t = 0$

In this section, we compute the Laurent series of (x, y, z, w) at $t = 0$ for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$. We can prove the following proposition in the same way as discussions in Section 1.

Proposition 2.1. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a meromorphic solution at $t = 0$. One of the following then occurs.*
(1) *x, y, z, w are all holomorphic at $t = 0$;*
(2) *x has a pole of order one at $t = 0$ and y, z, w are all holomorphic at $t = 0$;*
(3) *z has a pole of order n $(n \ge 1)$ at $t = 0$ and x, y, w are all holomorphic at $t = 0$;*
(4) *x has a pole of order one at $t = 0$ and z has a pole of order n $(n \ge 1)$ at $t = 0$ and y, w are both holomorphic at $t = 0$;*
(5) *y, w both have a pole of order n $(n \ge 1)$ at $t = 0$ and x, z are both holomorphic at $t = 0$.*

In order to determine the meromorphic solutions at $t = 0$, we set

$$\begin{cases} x = a_{0,n_0} t^{n_0} + a_{0,n_0-1} t^{n_0-1} + \cdots + a_{0,n_0+k} t^{n_0+k} + \cdots, \\ y = b_{0,n_1} t^{n_1} + b_{0,n_1-1} t^{n_1-1} + \cdots + a_{0,n_1+k} t^{n_1+k} + \cdots, \\ z = c_{0,n_2} t^{n_2} + c_{0,n_2-1} t^{n_2-1} + \cdots + c_{0,n_2+k} t^{n_2+k} + \cdots, \\ w = d_{0,n_3} t^{n_3} + d_{0,n_3-1} t^{n_3-1} + \cdots + d_{0,n_3+k} t^{n_3+k} + \cdots, \end{cases}$$

where n_0, n_1, n_2, n_3 are all integers. Especially, we note that $a_{0,-1} = \mathrm{Res}_{t=0} x$ and the constant terms of y and w are given by $b_{0,0}$ and $d_{0,0}$.

The aim of this section is to show that $a_{0,-1} = -\alpha_0 + \alpha_1$ and $b_{0,0} + d_{0,0} = 0, -\alpha_4 + \alpha_5$.

2.1 The case where x, y, z, w are all holomorphic at $t = 0$

In this subsection, we treat the case where x, y, z, w are all holomorphic at $t = 0$.

Proposition 2.2. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution such that x, y, z, w are all holomorphic at $t = 0$. Then,*

$$b_{0,0} + d_{0,0} = 0 \ \textit{or} \ -\alpha_4 + \alpha_5.$$

Proof. By comparing the constant terms in

$$\begin{cases} tx' = 2x^2y + tx^2 - 2xy - \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\} x \\ \qquad\qquad\qquad\qquad\qquad\qquad + (\alpha_2 + \alpha_5) + 2z\{(z-1)w + \alpha_3\} \\ ty' = -2xy^2 + y^2 - 2txy + \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\} y - \alpha_1 t \\ tz' = 2z^2w + tz^2 - 2zw - \{t + (\alpha_5 + \alpha_4)\} z + \alpha_5 + 2yz(z-1) \\ tw' = -2zw^2 + w^2 - 2tzw + \{t + (\alpha_5 + \alpha_4)\} w - \alpha_3 t - 2y(-w + 2zw + \alpha_3), \end{cases}$$

we have

$$\begin{aligned} &2a_{0,0}^2 b_{0,0} - 2a_{0,0}b_{0,0} - (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)a_{0,0} \\ &\qquad\qquad + (\alpha_2 + \alpha_5) + 2c_{0,0}(c_{0,0} - 1)d_{0,0} + 2\alpha_3 c_{0,0} = 0, \qquad (2.1) \\ &-2a_{0,0}b_{0,0}^2 + b_{0,0}^2 + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)b_{0,0} = 0, \qquad (2.2) \\ &2c_{0,0}^2 d_{0,0} - 2c_{0,0}d_{0,0} - (\alpha_5 + \alpha_4)c_{0,0} + \alpha_5 + 2b_{0,0}c_{0,0}(c_{0,0} - 1) = 0, \qquad (2.3) \\ &-2c_{0,0}d_{0,0}^2 + d_{0,0}^2 + (\alpha_5 + \alpha_4)d_{0,0} + 2b_{0,0}d_{0,0} - 4b_{0,0}c_{0,0}d_{0,0} - 2\alpha_3 b_{0,0} = 0, \qquad (2.4) \end{aligned}$$

respectively. From (2.1) and (2.2), we get

$$-a_{0,0}b_{0,0}^2 + (\alpha_2 + \alpha_5)b_{0,0} + 2b_{0,0}c_{0,0}(c_{0,0} - 1)d_{0,0} + 2\alpha_3 b_{0,0}c_{0,0} = 0. \qquad (2.5)$$

From (2.3) and (2.4), we obtain

$$-c_{0,0}d_{0,0}^2 + \alpha_5 d_{0,0} - 2b_{0,0}c_{0,0}^2 d_{0,0} - 2\alpha_3 b_{0,0}c_{0,0} = 0. \qquad (2.6)$$

Equations (2.5) and (2.6) imply that

$$-a_{0,0}b_{0,0}^2 + (\alpha_2 + \alpha_5)b_{0,0} - 2b_{0,0}c_{0,0}d_{0,0} + 2\alpha_3 b_{0,0}c_{0,0} - c_{0,0}d_{0,0}^2 + \alpha_5 d_{0,0} - 2\alpha_3 b_{0,0}c_{0,0} = 0. \qquad (2.7)$$

From (2.2), we have

$$a_{0,0}b_{0,0}^2 = \frac{1}{2}b_{0,0}^2 + \left(\alpha_2 + \alpha_3 + \frac{1}{2}\alpha_5 + \frac{1}{2}\alpha_4\right) b_{0,0}. \tag{2.8}$$

From (2.7) and (2.8), we get

$$-\frac{1}{2}b_{0,0}^2 + \left(-\alpha_3 + \frac{1}{2}\alpha_5 - \frac{1}{2}\alpha_4\right) b_{0,0} - 2b_{0,0}c_{0,0}d_{0,0} - c_{0,0}d_{0,0}^2 + \alpha_5 d_{0,0} = 0. \tag{2.9}$$

By (2.4), we obtain

$$2b_{0,0}c_{0,0}d_{0,0} = -c_{0,0}d_{0,0}^2 + \frac{1}{2}d_{0,0}^2 + \left(\frac{1}{2}\alpha_5 + \frac{1}{2}\alpha_4\right) d_{0,0} + b_{0,0}d_{0,0} - \alpha_3 b_{0,0}. \tag{2.10}$$

From (2.9) and (2.10), we find that

$$-\frac{1}{2}(b_{0,0} + d_{0,0})(b_{0,0} + d_{0,0} + \alpha_4 - \alpha_5) = 0, \tag{2.11}$$

which implies that

$$b_{0,0} + d_{0,0} = 0 \text{ or } -\alpha_4 + \alpha_5.$$

□

2.1.1 The case where $b_{0,0} + d_{0,0} = 0$

Let us treat the case where $b_{0,0} + d_{0,0} = 0$ and $b_{0,0} = 0$.

Proposition 2.3. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution such that x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = 0$, $b_{0,0} = 0$. One of the following then occurs:*
(1) $\alpha_0 = \alpha_1 = \alpha_4 = \alpha_5 = 0$,
(2) $\alpha_0 + \alpha_1 \neq 0$ *and* $\alpha_4 = \alpha_5 = 0$ *and*

$$b_{0,1} = \frac{-\alpha_1}{\alpha_0 + \alpha_1}, \; d_{0,1} = \frac{\alpha_3(-\alpha_0 + \alpha_1)}{\alpha_0 + \alpha_1},$$

(3) $\alpha_0 = \alpha_1 = 0$ *and* $\alpha_4 + \alpha_5 \neq 0$ *and*

$$a_{0,0} = \frac{\alpha_5}{\alpha_4 + \alpha_5} + \frac{\alpha_2(\alpha_4 - \alpha_5)}{\alpha_4 + \alpha_5}, \; c_{0,0} = \frac{\alpha_5}{\alpha_4 + \alpha_5},$$

(4) $\alpha_0+\alpha_1 \neq 0$ *and* $\alpha_4+\alpha_5 \neq 0$,

$$b_{0,1}=\frac{-\alpha_1}{\alpha_0+\alpha_1},\ c_{0,0}=\frac{\alpha_5}{\alpha_4+\alpha_5}.$$

and one of the following occurs:

(i) *If* $\alpha_0+\alpha_1=1$ *and* $\alpha_4+\alpha_5=1$, *one of the following occurs.*

(a) $\alpha_0=\alpha_1=1/2,\ \alpha_2=0,\ \alpha_3=-1/2,$ (b) $\alpha_2=-1/2,\ \alpha_3=0,\ \alpha_4=\alpha_5=1/2,$

(c) $\alpha_0=\alpha_1=1/2,\ \alpha_4=\alpha_5=1/2.$

(ii) *If* $\alpha_0+\alpha_1 \neq 1$ *and* $\alpha_4+\alpha_5=1$,

$$\alpha_3=0, \textit{ or } \alpha_0=\alpha_1,$$

and

$$a_{0,0}=\alpha_5+\frac{\alpha_2(-\alpha_4+\alpha_5)}{\alpha_0+\alpha_1-1},\ c_{0,0}=\alpha_5.$$

(iii) *If* $\alpha_0+\alpha_1=1$ *and* $\alpha_4+\alpha_5 \neq 1$,

$$\alpha_2=0, \textit{ or } \alpha_4=\alpha_5$$

and

$$b_{0,1}=-\alpha_1,\ d_{0,1}=\frac{\alpha_3(\alpha_0-\alpha_1)}{\alpha_4+\alpha_5-1}.$$

(iv) *If* $\alpha_0+\alpha_1 \neq 1$ *and* $\alpha_4+\alpha_5 \neq 1$,

$$\begin{cases} a_{0,0}=\dfrac{\alpha_5}{\alpha_4+\alpha_5}+\dfrac{\alpha_2(-\alpha_4+\alpha_5)}{(\alpha_4+\alpha_5)(\alpha_0+\alpha_1-1)},\ b_{0,1}=\dfrac{-\alpha_1}{\alpha_0+\alpha_1}, \\ c_{0,0}=\dfrac{\alpha_5}{\alpha_4+\alpha_5},\ d_{0,1}=\dfrac{\alpha_3(\alpha_0-\alpha_1)}{(\alpha_0+\alpha_1)(\alpha_4+\alpha_5-1)}. \end{cases}$$

Let us deal with the case where $b_{0,0}+d_{0,0}=0$ and $b_{0,0} \neq 0$.

Proposition 2.4. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, *there exists a solution such that* x,y,z,w *are all holomorphic at* $t=0$ *and* $b_{0,0}+d_{0,0}=0$ *and* $b_{0,0}\neq 0$. *Then,*

$$a_{0,0}=\frac{1}{2}+\frac{1}{2}(2\alpha_2+2\alpha_3+\alpha_4+\alpha_5)\frac{1}{b_{0,0}},\ c_{0,0}=\frac{1}{2}+\frac{1}{2}(2\alpha_3+\alpha_4+\alpha_5)\frac{1}{b_{0,0}},\ d_{0,0}=-b_{0,0}.$$

Moreover, one of the following occurs:

(1) $\alpha_4=\alpha_5=0$,

(2) $c_{0,0}=1/2$ *and* $-\alpha_4+\alpha_5=0$ *and* $\alpha_3+\alpha_4=0$,

(3) $c_{0,0}=\alpha_5/(\alpha_4+\alpha_5)\neq 1/2$, *and*

$$\begin{cases} a_{0,0}=\dfrac{\alpha_5}{\alpha_4+\alpha_5}+\dfrac{\alpha_2(-\alpha_4+\alpha_5)}{(\alpha_4+\alpha_5)(2\alpha_3+\alpha_4+\alpha_5)},\ b_{0,0}=\dfrac{(\alpha_4+\alpha_5)(2\alpha_3+\alpha_4+\alpha_5)}{-\alpha_4+\alpha_5}, \\ c_{0,0}=\dfrac{\alpha_5}{\alpha_4+\alpha_5},\ d_{0,0}=-\dfrac{(\alpha_4+\alpha_5)(2\alpha_3+\alpha_4+\alpha_5)}{-\alpha_4+\alpha_5}. \end{cases}$$

2.1.2 The case where $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5 \neq 0$

Now, let us suppose that $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5$.

Proposition 2.5. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5 \neq 0$. Then,*

$$c_{0,0} = \frac{1}{2}, \ \ \textit{or} \ \ \frac{\alpha_5}{-\alpha_4 + \alpha_5}.$$

Proof. From (2.3), it follows that

$$2(-\alpha_4 + \alpha_5)c_{0,0}^2 + (\alpha_4 - 3\alpha_5)c_{0,0} + \alpha_5 = 0.$$

By solving this quadratic equation with respect to $c_{0,0}$, we have

$$c_{0,0} = \frac{1}{2}, \ \ or \ \ \frac{\alpha_5}{-\alpha_4 + \alpha_5}.$$

□

Let us treat the case where $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5 \neq 0$ and $c_{0,0} = 1/2$.

Proposition 2.6. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that all of (x, y, z, w) are holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5 \neq 0$ and $c_{0,0} = 1/2$. One of the following then occurs:*
(1) $\alpha_3 = 0, \ \alpha_4 + \alpha_5 = 0,$
(2) $2\alpha_3 + \alpha_4 + \alpha_5 \neq 0$ *and* $\alpha_4 + \alpha_5 = 0$ *and*

$$b_{0,0} = 0, \ d_{0,0} = -\alpha_4 + \alpha_5,$$

(3) $2\alpha_3 + \alpha_4 + \alpha_5 \neq 0$ *and*

$$\begin{aligned} a_{0,0} &= \frac{1}{2} + \frac{(\alpha_0 + \alpha_1 - 1)(\alpha_0 + \alpha_1 + 2\alpha_2 - 1)}{2(-\alpha_4 + \alpha_5)(\alpha_4 + \alpha_5)}, \ b_{0,0} = \frac{(-\alpha_4 + \alpha_5)(\alpha_4 + \alpha_5)}{2\alpha_3 + \alpha_4 + \alpha_5} \neq 0, \\ c_{0,0} &= \frac{1}{2}, \ d_{0,0} = \frac{2\alpha_3(-\alpha_4 + \alpha_5)}{2\alpha_3 + \alpha_4 + \alpha_5}. \end{aligned}$$

Let us deal with the case where $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5 \neq 0$ and $c_{0,0} = \alpha_5/(-\alpha_4 + \alpha_5) \neq 1/2$, which implies that $\alpha_4 + \alpha_5 \neq 0$.

Proposition 2.7. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, *there exists a solution such that* x, y, z, w *are all holomorphic at* $t = 0$ *and* $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5 \neq 0$ *and* $c_{0,0} = \alpha_5/(-\alpha_4 + \alpha_5) \neq 1/2$, *which implies that* $\alpha_4 + \alpha_5 \neq 0$. *One of the following then occurs:*

(1) $\alpha_0 + \alpha_1 = 1$ *and* $\alpha_2 = 0$.

$$b_{0,0} = 0,\ d_{0,0} = -\alpha_4 + \alpha_5,$$

(2) $\alpha_0 + \alpha_1 \neq 1$, *and*

$$\begin{cases} a_{0,0} = \dfrac{\alpha_5}{-\alpha_4 + \alpha_5} + \dfrac{\alpha_2(\alpha_4 + \alpha_5)}{(\alpha_0 + \alpha_1 - 1)(-\alpha_4 + \alpha_5)},\ b_{0,0} = 0, \\ c_{0,0} = \dfrac{\alpha_5}{-\alpha_4 + \alpha_5},\ d_{0,0} = -\alpha_4 + \alpha_5, \end{cases}$$

(3) $b_{0,0} \neq 0$ *and*

$$\begin{cases} a_{0,0} = \dfrac{\alpha_5}{-\alpha_4 + \alpha_5} + \dfrac{\alpha_2(\alpha_4 + \alpha_5)}{(-\alpha_4 + \alpha_5)(2\alpha_3 + \alpha_4 + \alpha_5)},\ b_{0,0} = \dfrac{(-\alpha_4 + \alpha_5)(2\alpha_3 + \alpha_4 + \alpha_5)}{\alpha_4 + \alpha_5}, \\ c_{0,0} = \dfrac{\alpha_5}{-\alpha_4 + \alpha_5},\ d_{0,0} = -\dfrac{2\alpha_3(-\alpha_4 + \alpha_5)}{\alpha_4 + \alpha_5}. \end{cases}$$

Proof. From (2.4), it follows that

$$(2c_{0,0} - 1)d_{0,0}^2 - (2\alpha_3 - \alpha_4 - \alpha_5)d_{0,0} - 2\alpha_3(-\alpha_4 + \alpha_5) = 0.$$

By solving this equation with respect to $d_{0,0}$, we have

$$d_{0,0} = -\alpha_4 + \alpha_5,\ -\frac{2\alpha_3(-\alpha_4 + \alpha_5)}{\alpha_4 + \alpha_5}.$$

If $d_{0,0} = -\alpha_4 + \alpha_5$, we get $b_{0,0} = 0$. If $d_{0,0} = -\dfrac{2\alpha_3(-\alpha_4 + \alpha_5)}{\alpha_4 + \alpha_5}$, we obtain

$$b_{0,0} = \frac{(-\alpha_4 + \alpha_5)(2\alpha_3 + \alpha_4 + \alpha_5)}{\alpha_4 + \alpha_5},$$

because $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5$.

We first suppose that $b_{0,0} = 0$, which implies that $d_{0,0} = -\alpha_4 + \alpha_5$. From (2.1), it follows that

$$(1 - \alpha_0 - \alpha_1)a_{0,0} = \frac{-\alpha_2(\alpha_4 + \alpha_5) + \alpha_5(1 - \alpha_0 - \alpha_1)}{-\alpha_4 + \alpha_5},$$

where $\alpha_4+\alpha_5\neq 0$, because $c_{0,0}=\dfrac{\alpha_5}{-\alpha_4+\alpha_5}\neq\dfrac{1}{2}$. If $\alpha_0+\alpha_1=1$, then $\alpha_2=0$. If $\alpha_0+\alpha_1\neq 1$, we have

$$a_{0,0}=\frac{\alpha_5}{-\alpha_4+\alpha_5}+\frac{\alpha_2(\alpha_4+\alpha_5)}{(\alpha_0+\alpha_1-1)(-\alpha_4+\alpha_5)}.$$

We assume that

$$b_{0,0}=\frac{(-\alpha_4+\alpha_5)(2\alpha_3+\alpha_4+\alpha_5)}{\alpha_4+\alpha_5}\neq 0,\ \ d_{0,0}=-\frac{2\alpha_3(-\alpha_4+\alpha_5)}{\alpha_4+\alpha_5}.$$

From (2.2), it follows that

$$\begin{aligned}a_{0,0}&=\frac{1}{2}+\frac{1}{2}(2\alpha_2+2\alpha_3+\alpha_4+\alpha_5)\frac{1}{b_{0,0}}\\&=\frac{\alpha_5}{-\alpha_4+\alpha_5}+\frac{\alpha_2(\alpha_4+\alpha_5)}{(-\alpha_4+\alpha_5)(2\alpha_3+\alpha_4+\alpha_5)}.\end{aligned}$$

□

2.2 The case where x has a pole of order one at $t=0$

In this subsection, we treat the case where x has a pole of order one at $t=0$.

Proposition 2.8. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that x has a pole of order one at $t=0$ and y,z,w are all holomorphic at $t=0$. Then, $a_{0,-1}=-\alpha_0+\alpha_1$ and $b_{0,0}=0,\ b_{0,1}=\alpha_1/(\alpha_0-\alpha_1)$. Moreover, one of the following occurs:*
(1) $(\alpha_4+\alpha_5)c_{0,0}=\alpha_5$ *and* $d_{0,0}=0$,
(2) $c_{0,0}=\alpha_5/(-\alpha_4+\alpha_5)$ *and* $d_{0,0}=-\alpha_4+\alpha_5\neq 0$.

Proof. It can be proved by direct calculation. □

Corollary 2.9. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that x has a pole of order one at $t=0$ and y,z,w are all holomorphic at $t=0$. Moreover, assume that $\alpha_0\neq 0$ or $\alpha_1\neq 0$. s_0 or s_1 then transforms the solution into a solution such that x,y,z,w are all holomorphic at $t=0$.*

2.3 The case where z has a pole at $t=0$

Let us treat the case where z has a pole of order n $(n\geq 1)$ at $t=0$.

Proposition 2.10. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that z has a pole of order n $(n\geq 1)$ at $t=0$ and x,y,w are all holomorphic at $t=0$.*
(1) *If $n\geq 2$, then $\alpha_0=\alpha_1$ and*

$$\begin{cases} x=a_{0,0}+\cdots\ , \\ y=-\dfrac{1}{2}t+b_{0,n}t^n+\cdots\ , \\ z=c_{0,-n}t^{-n}+\cdots\ , \\ w=-\dfrac{\alpha_3}{c_{0,-n}}t^n+\cdots\ , \end{cases}$$

where $c_{0,-n}$ is not zero and $a_{0,0}$ is unknown.
(2) *If $n=1$, then*

$$\begin{cases} x=a_{0,0}+\cdots\ , \\ y=-\left(\dfrac{\alpha_0+\alpha_1+2\alpha_2}{2c_{0,-1}}+\dfrac{1}{2}\right)t+\cdots\ , \\ z=c_{0,-1}t^{-1}+\cdots\ , \\ w=-\dfrac{\alpha_3}{c_{0,-1}}t+\cdots\ , \end{cases}$$

where $c_{0,-1}$ is not zero and satisfies

$$(-\alpha_0+\alpha_1)c_{0,-1}=(\alpha_0+\alpha_1)(\alpha_0+\alpha_1+2\alpha_2).$$

Proof. It can be proved by direct calculation. □

Corollary 2.11. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ there exists a solution such that z has a pole of order n $(n\geq 1)$ at $t=0$ and x,y,w are all holomorphic at $t=0$. Moreover, assume that $\alpha_3\neq 0$. s_3 then transforms the solution into a solution such that x,y,z,w are all holomorphic at $t=0$.*

Proof. By direct calculation, we find that $s_3(x,y,z,w)$ is a solution of $D_5^{(1)}(\alpha_0,\alpha_1,\alpha_2+\alpha_3,-\alpha_3,\alpha_4+\alpha_3,\alpha_5+\alpha_3)$ such that $s_3(z)$ has a pole of at most order $n-1$ at $t=0$ and all of $s_3(x,y,w)$ are holomorphic at $t=0$. Moreover, we observe that $s_3(w)=(-\alpha_3/c_{0,-n})t^n+\cdots$.

If $n=1$, the corollary is proved. Now, let us suppose that $n\geq 2$ and $s_3(z)$ is not holomorphic at $t=0$. It then follows from Proposition 2.10 that

$$s_3(z)=c_{0,-m}t^{-m}+\cdots\ ,\ s_3(w)=-\frac{(-\alpha_3)}{c_{0,-m}}t^m+\cdots\ ,\ (1\leq m\leq n-1),$$

which is contradiction. □

2.4 The case where x, z have a pole at $t = 0$

In this subsection, we treat the case where x has a pole of order one at $t = 0$ and z has a pole of order n $(n \geq 1)$ at $t = 0$.

2.4.1 The case where z has a pole of order n $(n \geq 2)$ at $t = 0$

Proposition 2.12. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that x has a pole of order one at $t = 0$ and z has a pole of order n $(n \geq 2)$ at $t = 0$ and y, w are both holomorphic at $t = 0$. Then,*

$$\begin{cases} x = (-\alpha_0 + \alpha_1)t^{-1} + \cdots, \\ y = -\dfrac{1}{2}t + b_{0,n}t^n + \cdots, \\ z = c_{0,-n}t^{-n} + \cdots, \\ w = -\dfrac{\alpha_3}{c_{0,-n}}t^n + \cdots. \end{cases}$$

Proof. It can be proved by direct calculation. □

Corollary 2.13. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that x has a pole of order one at $t = 0$ and z has a pole of order n $(n \geq 2)$ at $t = 0$ and y, w are both holomorphic at $t = 0$. Moreover, assume that $\alpha_3 \neq 0$. s_3 then transforms the solution into a solution such that one of the following occurs:*
(1) x has a pole of order one at $t = 0$ and y, z, w are all holomorphic at $t = 0$;
(2) x, z both have a pole of order one at $t = 0$ and y, w are both holomorphic at $t = 0$.

Proof. By direct calculation, we find that $s_3(x, y, z, w)$ is a solution of $D_5^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5 + \alpha_3)$ such that $s_3(x)$ has a pole of order one at $t = 0$ and $s_3(z)$ has a pole of order m, $(0 \leq m \leq n - 1)$ at $t = 0$ and both of $s_3(y, w)$ are holomorphic at $t = 0$. Moreover, we observe that

$$s_3(z) = c'_{0,-m}t^{-m} + \cdots, \ s_3(w) = -\frac{\alpha_3}{c_{0,-n}}t^n + \cdots.$$

If $n = 1, 2$, the corollary is then proved. Now, let us suppose that $n \geq 3$ and $s_3(z)$ has a pole of order m $(2 \leq m \leq n - 1)$ at $t = 0$. It then follows from Proposition 2.12 that

$$s_3(w) = -\frac{(-\alpha_3)}{c'_{0,-m}}t^m + \cdots,$$

which is contradiction. □

2.4.2 The case where z has a pole of order one at $t=0$

Proposition 2.14. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that x, z both have a pole of order one at $t=0$ and y, w are both holomorphic at $t=0$. Then, $a_{0,-1}=-\alpha_0+\alpha_1$ and $b_{0,0}=d_{0,0}=0$ and $b_{0,1}=-1/2,\ \alpha_1/(\alpha_0-\alpha_1)$.*
(1) If $b_{0,1}=-1/2$, then $d_{0,1}=(-1+\alpha_4+\alpha_5)/2c_{0,-1}$, where $c_{0,-1}\neq 0$ satisfies the following condition:

$$(\alpha_0+\alpha_1+2\alpha_2)c_{0,-1}=(\alpha_0+\alpha_1)(-\alpha_0+\alpha_1). \tag{2.12}$$

(2) If $b_{0,1}=\alpha_1/(\alpha_0-\alpha_1)\neq -1/2$, then

$$d_{0,1}=-\frac{1}{2}-\frac{\alpha_1}{\alpha_0-\alpha_1}+\frac{-1+\alpha_4+\alpha_5}{2c_{0,-1}},$$

where $c_{0,-1}\neq 0$ satisfies

$$(\alpha_0+\alpha_1)c_{0,-1}=(\alpha_0+\alpha_1+2\alpha_2)(-\alpha_0+\alpha_1).$$

Proof. By comparing the coefficients of the term t^{-2} in

$$\begin{aligned} tx' = 2x^2y+tx^2-2xy-\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}\,x \\ +(\alpha_2+\alpha_5)+2z\{(z-1)w+\alpha_3\}, \end{aligned}$$

we have

$$2a_{0,-1}^2b_{0,0}+2c_{0,-1}^2d_{0,0}=0. \tag{2.13}$$

By comparing the coefficients of the term t^{-1} in

$$ty'=-2xy^2+y^2-2txy+\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}\,y-\alpha_1 t,$$

we get $-2a_{0,-1}b_{0,0}=0$, which implies that $b_{0,0}=d_{0,0}=0$.

By comparing the coefficients of the term t^{-1} in

$$\begin{aligned} tx' = 2x^2y+tx^2-2xy-\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}\,x \\ +(\alpha_2+\alpha_5)+2z\{(z-1)w+\alpha_3\}, \end{aligned}$$

we have

$$-(\alpha_0+\alpha_1)a_{0,-1}=2a_{0,-1}^2b_{0,1}+a_{0,-1}^2+2c_{0,-1}^2d_{0,1}+2\alpha_3c_{0,-1}. \tag{2.14}$$

By comparing the coefficients of the term t in

$$ty'=-2xy^2+y^2-2txy+\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}\,y-\alpha_1 t,$$

we get

$$(\alpha_0+\alpha_1)b_{0,1} = -2a_{0,-1}b_{0,1}^2 - 2a_{0,-1}b_{0,1} - \alpha_1. \tag{2.15}$$

From (2.14) and (2.15), we obtain

$$-a_{0,-1}^2 b_{0,1} + 2c_{0,-1}^2 b_{0,1} d_{0,1} + 2\alpha_3 b_{0,1} c_{0,-1} - \alpha_1 a_{0,-1}. \tag{2.16}$$

By comparing the coefficients of the term t^{-1} in

$$tz' = 2z^2 w + tz^2 - 2zw - \{t + (\alpha_5+\alpha_4)\} z + \alpha_5 + 2yz(z-1),$$

we have

$$-1 + \alpha_4 + \alpha_5 = 2c_{0,-1}d_{0,1} + c_{0,-1} + 2b_{0,1}c_{0,-1}. \tag{2.17}$$

By comparing the coefficients of the term t in

$$tw' = -2zw^2 + w^2 - 2tzw + \{t + (\alpha_5+\alpha_4)\} w - \alpha_3 t - 2y(-w + 2zw + \alpha_3),$$

we get

$$-(-1+\alpha_4+\alpha_5)d_{0,1} = -2c_{0,-1}d_{0,1}^2 - 2c_{0,-1}d_{0,1} - \alpha_3 - 4b_{0,1}c_{0,-1}d_{0,1} - 2\alpha_3 b_{0,1}. \tag{2.18}$$

From (2.17) and (2.18), we obtain

$$-c_{0,-1}d_{0,1} - \alpha_3 - 2b_{0,1}c_{0,-1}d_{0,1} - 2\alpha_3 b_{0,1} = 0. \tag{2.19}$$

The equations (2.16) and (2.19) implies that

$$a_{0,-1}^2 b_{0,1} + \alpha_1 a_{0,-1} = -c_{0,-1}^2 d_{0,1} - \alpha_3 c_{0,-1}. \tag{2.20}$$

Thus, it follows from (2.14) and (2.20) that $a_{0,-1} = -\alpha_0 + \alpha_1$. Therefore, it follows from (2.15) that

$$2(-\alpha_0+\alpha_1)b_{0,1}^2 + (-\alpha_0 + 3\alpha_1)b_{0,1} + \alpha_1 = 0, \tag{2.21}$$

which implies that $b_{0,1} = -1/2,\ \alpha_1/(\alpha_0-\alpha_1)$.

If $b_{0,1} = -1/2$, it follows from (2.14) and (2.17) that

$$(\alpha_0+\alpha_1+2\alpha_2)c_{0,-1} = (\alpha_0+\alpha_1)(-\alpha_0+\alpha_1) \tag{2.22}$$

and

$$d_{0,1} = \frac{-1+\alpha_4+\alpha_5}{2c_{0,-1}}. \tag{2.23}$$

If $b_{0,1} = \alpha_1/(\alpha_0-\alpha_1)$, it follows from (2.14) and (2.17) that

$$(\alpha_0+\alpha_1)c_{0,-1} = (\alpha_0+\alpha_1+2\alpha_2)(-\alpha_0+\alpha_1)$$

and

$$d_{0,1} = -\frac{1}{2} - \frac{\alpha_1}{\alpha_0-\alpha_1} + \frac{-1+\alpha_4+\alpha_5}{2c_{0,-1}}.$$

□

Lemma 2.15. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that $x \equiv z$. Then, $x \equiv z = 1/2$ or $\alpha_2 = 0$.*

Proof. It can be proved by direct calculation. □

Corollary 2.16. *If x, z both have a pole of order one at $t = 0$ and $\alpha_2 \neq 0$, then*

$$Res_{t=0}x \neq Res_{t=0}z.$$

Proof. From Lemma 2.15, it follows that $x \not\equiv z$, because x, z both have a pole at $t = 0$.

We first suppose that $a_{0,-1} = c_{0,-1}$. Then, $a_{0,0} \neq c_{0,0}$. For, if $a_{0,0} = c_{0,0}$, all of $s_2(x, y, z, w)$ have a pole at $t = 0$, which contradicts Proposition 2.1.

Therefore, $s_2(x), s_2(z)$ both have a pole of order one at $t = 0$ and $s_2(y), s_2(w)$ are both holomorphic at $t = 0$. Furthermore, the constant terms of the Taylor series of $s_2(y), s_2(w)$ are not zero, which is impossible from Proposition 2.14. □

2.4.3 Necessary conditions

Proposition 2.17. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that x has a pole of order one at $t = 0$ and z has a pole of order n $(n \geq 2)$ at $t = 0$ and y, w are both holomorphic at $t = 0$. One of the following then occurs:*
(1) $\alpha_0 + \alpha_1 = 0$, (2) $2\alpha_3 + \alpha_4 + \alpha_5 = 1$, (3) $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$.

Proof. We first prove that if $\alpha_3 \neq 0$, $\alpha_0 + \alpha_1 = 0$ or $2\alpha_3 + \alpha_4 + \alpha_5 = 1$. Since $\alpha_3 \neq 0$, it follows from Corollary 2.13 that $s_3(x, y, z, w)$ is a solution of $D_5^{(1)}(\alpha_0, \alpha_1, \alpha_2+\alpha_3, -\alpha_3, \alpha_4+\alpha_3, \alpha_5+\alpha_3)$ such that either of the following occurs:
(i) $s_3(x)$ has a pole of order one at $t = 0$ and all of $s_3(y, z, w)$ are holomorphic at$t = 0$,
(ii) both of $s_3(x, z)$ have a pole of order one at $t = 0$ and both of $s_3(y, w)$ are holomorphic at $t = 0$.

If case (i) occurs, then $\alpha_0 + \alpha_1 = 0$, because for $s_3(x, y, z, w)$, $b_{0,1} = -1/2$. If case (ii) occurs, then $2\alpha_3 + \alpha_4 + \alpha_5 = 1$, because for $s_3(x, y, z, w)$, $b_{0,1} = -1/2$ and $d_{0,1} = 0$.

Let us suppose that $\alpha_3 = 0$. If $\alpha_4 \neq 0$ or $\alpha_5 \neq 0$, we use s_4 or s_5 in the same way and can obtain necessary conditions, respectively.

Let us suppose that $\alpha_3 = \alpha_4 = \alpha_5 = 0$. We can then assume that $\alpha_2 \neq 0$. Thus, we use s_2 in the same way and can obtain necessary conditions. □

2.5 The case where y, w have a pole at $t = 0$

In this subsection, we treat the case where y, w have a pole of order n $(n \geq 1)$ at $t = 0$. We can obtain necessary conditions for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ to have a meromorphic solution at $t = \infty$.

2.5.1 The Laurent series of (x, y, z, w) at $t = 0$

Proposition 2.18. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that y, w both have a pole of order n $(n \geq 1)$ at $t = 0$ and x, z are both holomorphic at $t = 0$. Then,*

$$\begin{cases} x = \dfrac{1}{2} + \dfrac{n + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5}{2b_{0,-n}} t^n + \cdots, \\ y = b_{0,-n}t^{-n} + \cdots + b_{0,-1}t^{-1} + b_{0,0} + \cdots, \\ z = \dfrac{1}{2} + \dfrac{n + 2\alpha_3 + \alpha_4 + \alpha_5}{2b_{0,-n}} t^n + \cdots, \\ w = d_{0,-n}t^{-n} + \cdots + d_{0,-1}t^{-1} + d_{0,0} + \cdots, \end{cases}$$

where $b_{0,-k} + d_{0,-k} = 0$ $(1 \leq k \leq n)$ and $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5$.

Corollary 2.19. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that y, w both have a pole of order n $(n \geq 1)$ at $t = 0$ and x, z are both holomorphic at $t = 0$. Moreover, assume that $\alpha_2 \neq 0$. s_2 then transforms the solution into a solution such that x, y, z, w are all holomorphic at $t = 0$.*

Proof. By direct calculations, we find that $s_2(x, y, z, w)$ is a solution of $D_5^{(1)}(\alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5)$ such that both of $s_2(y, w)$ have a pole of order m $(0 \leq m \leq n-1)$ at $t = 0$ and both of $s_2(x, z)$ are holomorphic at $t = 0$. Moreover, we can check that

$$s_2(x) - s_2(z) = \frac{\alpha_2}{b_{0,-n}} t^n + \cdots.$$

If $n = 1$, the corollary is obvious. We then assume that $n \geq 2$ and $1 \leq m \leq n - 1$. Therefore, it follows from Proposition 2.18 that

$$s_2(y) = b'_{0,-m}t^{-m} + \cdots, \quad s_2(x) - s_2(z) = \frac{\alpha_2}{b'_{0,-m}} t^m + \cdots,$$

which is impossible. Thus, the corollary is proved. ∎

2.5.2 Necessary conditions

In order to obtain necessary conditions, we prove the following lemma:

Lemma 2.20. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} = c_{0,0} = 1/2$ and $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5$. One of the following then occurs:* (1) $\alpha_0 + \alpha_1 = 1$; (2) $\alpha_4 + \alpha_5 = 0$; (3) $-\alpha_4 + \alpha_5 = 0$.

Proof. The lemma follows from Proposition 2.6. ∎

Proposition 2.21. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that y, w both have a pole of order n ($n \geq 1$) at $t = 0$ and x, z are both holomorphic at $t = 0$. One of the following then occurs:*
(1) $\alpha_0+\alpha_1+2\alpha_2 = 1$; (2) $\alpha_4+\alpha_5 = 0$; (3) $-\alpha_4+\alpha_5 = 0$; (4) $\alpha_0 = \alpha_1 = \alpha_2 = 0$.

Proof. We first prove that case (1), (2) or (3) occurs if $\alpha_2 \neq 0$. Since $\alpha_2 \neq 0$, $s_2(x, y, z, w)$ is a solution of $D_5^{(1)}(\alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5)$ such that all of $s_2(x, y, z, w)$ are holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5$ and $a_{0,0} = c_{0,0} = 1/2$. It then follows from Lemma 2.20 that $\alpha_0 + \alpha_1 + 2\alpha_2 = 1$, or $\alpha_4 + \alpha_5 = 0$ or $-\alpha_4 + \alpha_5 = 0$.

If $\alpha_2 = 0$, we can assume that $\alpha_0 \neq 0$ or $\alpha_1 \neq 0$. When $\alpha_0 \neq 0$ or $\alpha_1 \neq 0$, by s_0 or s_1, we can obtain the necessary conditions, respectively. □

2.6 Summary

Proposition 2.22. (1) *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution meromorphic at $t = 0$. Moreover, assume that x has a pole at $t = 0$. x then has a pole of order one at $t = 0$ and*

$$Res_{t=0}x = -\alpha_0 + \alpha_1.$$

(2) *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a meromorphic solution at $t = 0$. $y + w$ is then holomorphic at $t = 0$ and*

$$b_{0,0} + d_{0,0} = 0, -\alpha_4 + \alpha_5.$$

3 Meromorphic solutions at $t = c \in \mathbb{C}^*$

In this section, we treat the meromorphic solution at $t = c \in \mathbb{C}^*$. For this purpose, we set

$$\begin{cases} x = a_{c,n_0}T^{n_0} + a_{c,n_0+1}T^{n_0+1} + \cdots + a_{c,n_0+k}T^{n_0+k} + \cdots, \\ y = b_{c,n_1}T^{n_1} + b_{c,n_1+1}T^{n_1+1} + \cdots + b_{c,n_1+k}T^{n_1+k} + \cdots, \\ z = c_{c,n_0}T^{n_2} + c_{c,n_2+1}T^{n_2+1} + \cdots + c_{c,n_2+k}T^{n_2+k} + \cdots, \\ w = d_{c,n_0}T^{n_3} + d_{c,n_3+1}T^{n_3+1} + \cdots + d_{c,n_3+k}T^{n_3+k} + \cdots, \end{cases}$$

where $T := t - c$ and n_0, n_1, n_2, n_3 are all integers.

We can prove the following proposition in the same way as discussion in Section 1.

Proposition 3.1. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that some of x, y, z, w have a pole at $t = c \in \mathbb{C}^*$. One of the following then occurs:*
(1) *x has a pole of order one at $t = c$ and y, z, w are all holomorphic at $t = c$;*
(2) *y has a pole of order one at $t = c$ and x, z, w are all holomorphic at $t = c$;*

(3) *z has a pole of order n $(n \geq 1)$ and x, y, w are all holomorphic at $t = c$;*
(4) *w has a pole of order one at $t = c$ and x, y, z are all holomorphic at $t = c$;*
(5) *x, z both have a pole of order one at $t = c$ and y, w are both holomorphic at $t = c$;*
(6) *x, w both have a pole of order one at $t = c$ and y, z are both holomorphic at $t = c$;*
(7) *y, w both have a pole of order n $(n \geq 1)$ at $t = c$ and x, z are both holomorphic at $t = c$.*

3.1 The case where x has a pole at $t = c \in \mathbb{C}^*$

Let us treat the case where x has a pole of order one at $t = c \in \mathbb{C}^*$

Proposition 3.2. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that x has a pole of order one at $t = c \in \mathbb{C}^*$ and y, z, w are all holomorphic at $t = c$. Then, $b_{c,0} = 0$ or $-c$.*
(1) *If $b_{c,0} = 0$, then*

$$\begin{cases} x = -(t-c)^{-1} + + \cdots, \\ y = \alpha_1(t-c) + \cdots. \end{cases}$$

(2) *If $b_{c,0} = -c$,*

$$\begin{cases} x = (t-c)^{-1} + \dfrac{-\alpha_0 + \alpha_1 + c - 1}{2c} + \cdots, \\ y = -c + (-\alpha_0 - 1)(t-c) + \cdots. \end{cases}$$

3.2 The case where y has a pole at $t = c \in \mathbb{C}^*$

Proposition 3.3. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that y has a pole of order one at $t = c \in \mathbb{C}^*$ and x, z, w are all holomorphic at $t = c$. Then, $a_{c,0} = 0, 1$ and $c_{c,0} = 0, 1$.*
(1) *If $(a_{c,0}, c_{c,0}) = (0, 0)$,*

$$\begin{cases} x = \dfrac{-\alpha_2 - \alpha_5}{c}(t-c) + \cdots, \\ y = -c(t-c)^{-1} + \cdots, \\ z = O(t-c), \\ w = \alpha_3 + O(t-c). \end{cases}$$

(2) *If* $(a_{c,0}, c_{c,0}) = (0,1)$,

$$\begin{cases} x = \dfrac{-\alpha_2 - 2\alpha_3 - \alpha_5}{c}(t-c) + \cdots, \\ y = -c(t-c)^{-1} + \cdots, \\ z = 1 + O(t-c), \\ w = -\alpha_3 + O(t-c). \end{cases}$$

(3) *If* $(a_{c,0}, c_{c,0}) = (1,0)$,

$$\begin{cases} x = 1 + \dfrac{\alpha_2 + 2\alpha_3 + \alpha_4}{c}(t-c) + \cdots, \\ y = c(t-c)^{-1} + \cdots, \\ z = O(t-c), \\ w = \alpha_3 + O(t-c). \end{cases}$$

(4) *If* $(a_{c,0}, c_{c,0}) = (1,1)$,

$$\begin{cases} x = 1 + \dfrac{\alpha_2 + \alpha_4}{c}(t-c) + \cdots, \\ y = c(t-c)^{-1} + \cdots, \\ z = 1 + O(t-c), \\ w = -\alpha_3 + O(t-c). \end{cases}$$

3.3 The case where z has a pole at $t = c \in \mathbb{C}^*$

Proposition 3.4. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$*, there exists a solution such that* z *has a pole of order* n $(n \ge 1)$ *at* $t = c \in \mathbb{C}^*$ *and* x, y, w *are all holomorphic at* $t = c \in \mathbb{C}^*$.
(1) *If* $n \ge 2$,

$$b_{c,0} = -\frac{c}{2}, \text{ and } d_{c,0} = d_{c,1} = \cdots = d_{c,n-1} = 0, \; d_{c,n} = -\frac{\alpha_3}{c_{c,-n}}.$$

(2) *If* $n = 1$,

$$c_{c,-1} = -\frac{c}{2b_{c,0} + c}, \; b_{c,0} \ne -\frac{c}{2} \text{ and } d_{c,0} = 0, \; d_{c,1} = -\frac{\alpha_3}{c_{c,-1}}.$$

3.4 The case where w has a pole at $t = c$

Proposition 3.5. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$*, there exists a solution such that* w *has a pole of order one at* $t = c \in \mathbb{C}^*$ *and* x, y, z *are all holomorphic at* $t = c$*. Then,* $c_{c,0} = 0, 1$.

(1) *If* $c_{c,0} = 0$,

$$\begin{cases} z = -\dfrac{\alpha_5}{c}(t-c) + \cdots, \\ w = -c(t-c)^{-1} + \cdots. \end{cases}$$

(2) *If* $c_{c,0} = 1$,

$$\begin{cases} z = 1 + \dfrac{\alpha_4}{c}(t-c) + \cdots, \\ w = c(t-c)^{-1} + \cdots. \end{cases}$$

3.5 The case where x, z have a pole at $t = c$

Proposition 3.6. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$*, there exists a solution such that* x, z *both have a pole of order one at* $t = c \in \mathbb{C}^*$ *and* y, w *are both holomorphic at* $t = c$*. Then,* $(b_{c,0}, d_{c,0}) = (0,0), (0,-c), (-c,0), (-c,c)$.
(1) *If* $(b_{c,0}, d_{c,0}) = (0,0)$,

$$\begin{cases} x = -(t-c)^{-1} + \cdots, \\ y = \alpha_1(t-c) + \cdots, \\ z = -(t-c)^{-1} + \cdots, \\ w = \alpha_3(t-c) + \cdots. \end{cases}$$

(2) *Assume that* $(b_{c,0}, d_{c,0}) = (0,-c)$*. Then,* $(a_{c,-1}, c_{c,-1}) = (-2,1), (1,1)$.
(i) *If* $(a_{c,-1}, c_{c,-1}) = (-2,1)$,

$$\begin{cases} x = -2(t-c)^{-1} + \cdots, \\ y = \dfrac{\alpha_1}{3}(t-c) + \cdots, \\ z = (t-c)^{-1} + \cdots, \\ w = -c + \left\{-2 + (\alpha_3 + \alpha_4 + \alpha_5) - \dfrac{4}{3}\alpha_1\right\}(t-c) + \cdots. \end{cases}$$

(ii) *If* $(a_{c,-1}, c_{c,-1}) = (1,1)$,

$$\begin{cases} x = (t-c)^{-1} + \cdots, \\ y = -\dfrac{\alpha_1}{3}(t-c) + \cdots, \\ z = (t-c)^{-1} + \cdots, \\ w = -c + \left\{-2 + (\alpha_3 + \alpha_4 + \alpha_5) + \dfrac{4}{3}\alpha_1\right\}(t-c) + \cdots. \end{cases}$$

(3) *If* $(b_{c,0}, d_{c,0}) = (-c, 0)$,

$$\begin{cases} x = (t-c)^{-1} + \cdots, \\ y = -c + (-1-\alpha_0)(t-c) + \cdots, \\ z = (t-c)^{-1} + \cdots, \\ w = -\alpha_3(t-c) + \cdots. \end{cases}$$

(4) *Assume that* $(b_{c,0}, d_{c,0}) = (-c, c)$. *Then,* $(a_{c,-1}, c_{c,-1}) = (2, -1), (-1, -1)$.
(i) *If* $(a_{c,-1}, c_{c,-1}) = (2, -1)$,

$$\begin{cases} x = 2(t-c)^{-1} + \cdots, \\ y = -c + \left\{-1 - \dfrac{1}{3}\alpha_0\right\}(t-c) + \cdots, \\ z = -(t-c)^{-1} + \cdots, \\ w = c + \left\{2 - (\alpha_3+\alpha_4+\alpha_5) + \dfrac{4}{3}\alpha_0\right\}(t-c) + \cdots. \end{cases}$$

(ii) *If* $(a_{c,-1}, c_{c,-1}) = (-1, -1)$,

$$\begin{cases} x = -(t-c)^{-1} + \cdots, \\ y = -c + \left\{-1 + \dfrac{1}{3}\alpha_0\right\}(t-c) + \cdots, \\ z = -(t-c)^{-1} + \cdots, \\ w = c + \left\{2 - (\alpha_3+\alpha_4+\alpha_5) - \dfrac{4}{3}\alpha_0\right\}(t-c) + \cdots. \end{cases}$$

3.6 The case where x, w have a pole at $t = c \in \mathbb{C}^*$

Let us treat the case where x, w both have a pole of order one at $t = c \in \mathbb{C}^*$ and y, z are both holomorphic at $t = c$.

Proposition 3.7. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, *there exists a solution such that* x, w *both have a pole of order one at* $t = c \in \mathbb{C}^*$ *and* y, z *are both holomorphic at* $t = c$. *Then,* $b_{c,0} = 0, -c$ *and* $c_{c,0} = 0, 1$.
(1) *If* $(b_{c,0}, c_{c,0}) = (0, 0)$,

$$\begin{cases} x = -(t-c)^{-1} + \cdots, \\ y = \alpha_1(t-c) + \cdots, \\ z = -\dfrac{\alpha_5}{c}(t-c) + \cdots, \\ w = -c(t-c)^{-1} + \cdots. \end{cases}$$

(2) *If* $(b_{c,0}, c_{c,0}) = (0, 1)$,

$$\begin{cases} x = -(t-c)^{-1} + \cdots, \\ y = \alpha_1 (t-c) + \cdots, \\ z = 1 + \dfrac{\alpha_4}{c}(t-c) + \cdots, \\ w = c(t-c)^{-1} + \cdots. \end{cases}$$

(3) *If* $(b_{c,0}, c_{c,0}) = (-c, 0)$,

$$\begin{cases} x = (t-c)^{-1} + \cdots, \\ y = -c + (-\alpha_0 - 1)(t-c) + \cdots, \\ z = -\dfrac{\alpha_5}{c}(t-c) + \cdots, \\ w = -c(t-c)^{-1} + \cdots. \end{cases}$$

(4) *If* $(b_{c,0}, c_{c,0}) = (-c, 1)$,

$$\begin{cases} x = (t-c)^{-1} + \cdots, \\ y = -c + (-\alpha_0 - 1)(t-c) + \cdots, \\ z = 1 + \dfrac{\alpha_4}{c}(t-c) + \cdots, \\ w = c(t-c)^{-1} + \cdots. \end{cases}$$

3.7 The case where y, w have a pole at $t = c$

Proposition 3.8. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution such that y, w both have a pole of order n $(n \ge 1)$ at $t = c \in \mathbb{C}^*$ and x, z are both holomorphic at $t = c$.*
(1) *Assume that $n \ge 2$. Then,*

$$\begin{aligned} &a_{c,0} = \frac{1}{2},\ a_{c,1} = \cdots = a_{c,n-2} = 0,\ a_{c,n-1} = \frac{nc}{2b_{c,-n}}, \\ &a_{c,n} = \frac{1}{2\{b_{c,-n}\}^2}\big[-(n+1)\{b_{c,-(n-1)}\}c + (n + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5) b_{c,-n} \big], \\ &c_{c,0} = \frac{1}{2},\ c_{c,1} = \cdots = c_{c,n-2} = 0,\ c_{c,n-1} = \frac{nc}{2b_{c,-n}}, \\ &c_{c,n} = \frac{1}{2\{b_{c,-n}\}^2}\big[-(n+1)\{b_{c,-(n-1)}\}c + (n + 2\alpha_3 + \alpha_4 + \alpha_5) b_{c,-n} \big], \\ &b_{c,-k} + d_{c,-k} = 0 \ (1 \le k \le n). \end{aligned}$$

(2) *Assume that* $n=1$. *Then,* $b_{c,-1}+d_{c,-1}=0$ *or* $c_{c,0}=0,1$.

(i) *If* $b_{c,-1}+d_{c,-1}=0$ *and* $c_{c,0}\neq 0,1$,

$$\begin{aligned}
&a_{c,0}=c_{c,0}=\frac{1}{2}+\frac{c}{2b_{c,-1}},\\
&a_{c,1}=\frac{1}{2\{b_{c,-1}\}^2}\left\{-2b_{c,0}c-c^2+(1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5)b_{c,-1}\right\},\\
&c_{c,1}=\frac{1}{2\{b_{c,-1}\}^2}\left\{-2b_{c,0}c-c^2+(1+2\alpha_3+\alpha_4+\alpha_5)b_{c,-1}\right\},\\
&b_{c,-1}+d_{c,-1}=0.
\end{aligned}$$

(ii) *Assume that* $n=1$ *and* $c_{c,0}=0,1$. *Then,* $a_{c,0}=0,1$.

(a) *If* $(a_{c,0},c_{c,0})=(0,0)$,

$$\begin{cases}
x=\dfrac{-\alpha_2+\alpha_5}{c}(t-c)+\cdots\,,\\
y=-c(t-c)^{-1}+\cdots\,,\\
z=\dfrac{\alpha_5}{c}(t-c)+\cdots\,,\\
w=c(t-c)^{-1}+\cdots\,.
\end{cases}$$

(b) *If* $(a_{c,0},c_{c,0})=(0,1)$,

$$\begin{cases}
x=\dfrac{-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5}{c}(t-c)+\cdots\,,\\
y=-c(t-c)^{-1}+\cdots\,,\\
z=1+\dfrac{\alpha_4}{3c}(t-c)+\cdots\,,\\
w=3c(t-c)^{-1}+\cdots\,.
\end{cases}$$

(c) *If* $(a_{c,0},c_{c,0})=(1,0)$,

$$\begin{cases}
x=1+\dfrac{\alpha_2+2\alpha_3+\alpha_4+2\alpha_5}{c}(t-c)+\cdots\,,\\
y=c(t-c)^{-1}+\cdots\,,\\
z=-\dfrac{\alpha_5}{3c}(t-c)+\cdots\,,\\
w=-3c(t-c)^{-1}+\cdots\,.
\end{cases}$$

(d) *If* $(a_{c,0}, c_{c,0}) = (1, 1)$,

$$\begin{cases} x = 1 + \dfrac{\alpha_2 - \alpha_4}{c}(t-c) + \cdots , \\ y = c(t-c)^{-1} + \cdots , \\ z = 1 - \dfrac{\alpha_4}{c}(t-c) + \cdots , \\ w = -c(t-c)^{-1} + \cdots . \end{cases}$$

3.8 Summary

Let us summarize the results in this section. Furthermore, by the discussions in Section 1 and 2, we can obtain a main tool which we use in order to show necessary conditions for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ to have a rational solution.

Proposition 3.9. (1) *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, *there exists a meromorphic solution at* $t = c \in \mathbb{C}^*$. *Moreover, assume that* x *has a pole at* $t = c$. x *then has pole of order one at* $t = c$ *and* $Res_{t=c}x \in \mathbb{Z}$.
(2) *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, *there exists a meromorphic solution at* $t = c \in \mathbb{C}^*$. $y + w$ *is then holomorphic at* $t = c$ *or has a pole of order one at* $t = c$. *If* $y + w$ *has a pole of order one at* $t = c$, $Res_{t=c}(y + w) = nc$, $(n \in \mathbb{Z})$.
(3) *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, *there exists a rational solution. Then,*

$$(b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}.$$

4 The Hamiltonian H and its properties

In this section, we compute the Laurent series of the Hamiltonian H at $t = \infty, 0, c \in \mathbb{C}^*$ for a meromorphic solution of $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ at $t = \infty, 0, c$, respectively. This section constitutes of five subsections.

In Subsection 4.1 and 4.2, we treat the Laurent series of H at $t = \infty$ for a solution of type A and type B, respectively. In Subsection 4.3 and 4.4, we deal with the Laurent series of H at $t = 0, c$, respectively. In Subsection 4.5, we explain the relationship between H and a rational solution of $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$.

4.1 The Hamiltonian for a solution of type A

In this subsection, we compute $h_{\infty,0}$ for a solution of type A of $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$.

4.1.1 The case where x, y, z, w are all holomorphic at $t = \infty$

Proposition 4.1. *Suppose that for $D_5^{(1)}(\alpha_j)_{\leq j \leq 5}$, there exists a solution of type A such that x, y, z, w are all holomorphic at $t = \infty$. Then, $a_{\infty,0} = 0, 1$ and $c_{\infty,0} = 0, 1$.*
(1) *If* $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$,

$$h_{\infty,0} = \alpha_1(\alpha_2 + \alpha_5) + \alpha_3\alpha_5.$$

(2) *If* $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$,

$$h_{\infty,0} = \alpha_1(\alpha_2 + 2\alpha_3 + \alpha_5) + \alpha_3\alpha_4.$$

(3) *If* $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$,

$$h_{\infty,0} = \alpha_1(\alpha_2 + 2\alpha_3 + \alpha_4) + \alpha_3\alpha_5.$$

(4) *If* $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$,

$$h_{\infty,0} = \alpha_1(\alpha_2 + \alpha_4) + \alpha_3\alpha_4.$$

4.1.2 The case where y has a pole at $t = \infty$

Proposition 4.2. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution of type A such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$. Then, $a_{\infty,0} = 0, 1$ and $c_{\infty,0} = 0, 1$.*
(1) *If* $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$,

$$h_{\infty,0} = -(1 - \alpha_0)(\alpha_2 + \alpha_5) + \alpha_3\alpha_5.$$

(2) *If* $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$,

$$h_{\infty,0} = -(1 - \alpha_0)(\alpha_2 + 2\alpha_3 + \alpha_5) + \alpha_3\alpha_4.$$

(3) *If* $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$,

$$h_{\infty,0} = -(1 - \alpha_0)(\alpha_2 + 2\alpha_3 + \alpha_4) + \alpha_5\alpha_3.$$

(4) *If* $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$,

$$h_{\infty,0} = -(1 - \alpha_0)(\alpha_2 + \alpha_4) + \alpha_4\alpha_3.$$

4.1.3 The case where w has a pole at $t = \infty$

Proposition 4.3. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution of type A such that w has a pole of order one at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$. Then, $a_{\infty,0} = 0, 1$ and $c_{\infty,0} = 0, 1$.*
(1) *If $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$,*

$$h_{\infty,0} = \alpha_1(\alpha_2 - \alpha_5) - \alpha_5(\alpha_3 + \alpha_4 + \alpha_5).$$

(2) *If $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$,*

$$h_{\infty,0} = \alpha_1(\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5) - \alpha_5(\alpha_3 + \alpha_4 + \alpha_5).$$

(3) *If $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$,*

$$h_{\infty,0} = \alpha_1(\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5) - \alpha_4(\alpha_3 + \alpha_4 + \alpha_5).$$

(4) *If $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$,*

$$h_{\infty,0} = \alpha_1(\alpha_2 - \alpha_4) - \alpha_4(\alpha_3 + \alpha_4 + \alpha_5).$$

4.1.4 The case where y, w have a pole at $t = \infty$

Proposition 4.4. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution of type A such that y, w both have a pole of order one at $t = \infty$ and x, z are both holomorphic at $t = \infty$. Then, $a_{\infty,0} = 0, 1$ and $c_{\infty,0} = 0, 1$.*
(1) *If $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$,*

$$h_{\infty,0} = (\alpha_0 - 1)(\alpha_2 - \alpha_5) - \alpha_5(\alpha_3 + \alpha_4 + \alpha_5).$$

(2) *If $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$,*

$$h_{\infty,0} = (\alpha_0 - 1)(\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5) - \alpha_4(\alpha_3 + \alpha_4 + \alpha_5).$$

(3) *If $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$,*

$$h_{\infty,0} = (\alpha_0 - 1)(\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5) - \alpha_5(\alpha_3 + \alpha_4 + \alpha_5).$$

(4) *If $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$,*

$$h_{\infty,0} = (\alpha_0 - 1)(\alpha_2 - \alpha_4) - \alpha_4(\alpha_3 + \alpha_4 + \alpha_5).$$

4.2 The Hamiltonian for a solution of type B

In this subsection, we calculate $h_{\infty,0}$ for a solution of type B of $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$.

4.2.1 The case where y has a pole at $t=\infty$

Proposition 4.5. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$. Then,*

$$h_{\infty,0}=\frac{1}{4}(-\alpha_4+\alpha_5)^2+\frac{1}{4}(-\alpha_0+\alpha_1)^2+\frac{1}{2}(-\alpha_0+\alpha_1).$$

Proposition 4.6. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$.*

(1) *Assume that case* (1) *occurs in Proposition 1.16. Then,*

$$h_{\infty,0}=\frac{1}{4}(2\alpha_3+\alpha_4+\alpha_5)^2+\frac{1}{4}(-\alpha_0+\alpha_1)^2+\frac{1}{2}(-\alpha_0+\alpha_1)-\alpha_4\alpha_5.$$

(2) *Assume that case* (2) *occurs in Proposition 1.16. Then,*

$$h_{\infty,0}=\frac{1}{4}(2\alpha_3+\alpha_4+\alpha_5)^2+\frac{1}{4}(-\alpha_0+\alpha_1)^2+\frac{1}{2}(-\alpha_0+\alpha_1).$$

(3) *Assume that case* (3) *occurs in Proposition 1.16. Then, $\alpha_4=\alpha_5=0$ and*

$$h_{\infty,0}=\alpha_3^2+\frac{1}{4}(-\alpha_0+\alpha_1)^2+\frac{1}{2}(-\alpha_0+\alpha_1).$$

4.2.2 The case where w has a pole at $t=\infty$

Proposition 4.7. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution of type B such that w has a pole at $t=\infty$ and x,y,z are all holomorphic at $t=\infty$. Then, $\alpha_1=0$ and*

$$h_{\infty,0}=\frac{1}{4}(-\alpha_4+\alpha_5)^2+\frac{1}{4}(\alpha_0+2\alpha_2)^2-\frac{1}{2}(\alpha_0+2\alpha_2).$$

4.2.3 The case where y,z have a pole at $t=\infty$

Proposition 4.8. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution of type B such that y has a pole of order one at $t=\infty$ and z has a pole of order n $(n\ge 1)$ at $t=\infty$ and x,w are both holomorphic at $t=\infty$. Then,*

$$h_{\infty,0}=\frac{1}{4}(-\alpha_0+\alpha_1)^2+\frac{1}{2}(-\alpha_0+\alpha_1)+\alpha_3(\alpha_3+\alpha_4+\alpha_5).$$

4.2.4 The case where y, w have a pole at $t = \infty$

Proposition 4.9. (1) *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution of type B such that y, w both have a pole of order n $(n \geq 2)$ at $t = \infty$ and x, z are both holomorphic at $t = \infty$. Then,*

$$h_{\infty,0} = \frac{1}{4}(-\alpha_4 + \alpha_5)^2 - \alpha_2(\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5).$$

(2) *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution of type B such that y, w both have a pole of order one at $t = \infty$ and x, z are both holomorphic at $t = \infty$.*
(i) *Assume that case* (1) *occurs in Proposition 1.41. Then,*

$$h_{\infty,0} = \frac{1}{4}(-\alpha_4 + \alpha_5)^2 + \frac{1}{4}(\alpha_0 + \alpha_1 + 2\alpha_2)^2 - \frac{1}{2}(\alpha_0 + \alpha_1 + 2\alpha_2) + \alpha_1(1 - \alpha_0).$$

(ii) *Assume that case* (2) *occurs in Proposition 1.41. Then,*

$$h_{\infty,0} = \frac{1}{4}(-\alpha_4 + \alpha_5)^2 + \frac{1}{4}(\alpha_0 + \alpha_1 + 2\alpha_2)^2 - \frac{1}{2}(\alpha_0 + \alpha_1 + 2\alpha_2).$$

(iii) *Assume that case* (3) *occurs in Proposition 1.41. Then,*

$$h_{\infty,0} = \frac{1}{4}(-\alpha_4 + \alpha_5)^2 + \alpha_2^2 - \alpha_2.$$

4.3 The Laurent series of H at $t = 0$

In this subsection, we compute the constant term $h_{0,0}$ of the Laurent series of H at $t = 0$ for a meromorphic solution of $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ at $t = 0$.

4.3.1 The case where x, y, z, w are holomorphic at $t = 0$

Proposition 4.10. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a solution such that x, y, z, w are all holomorphic at $t = 0$. Then, $b_{0,0} + d_{0,0} = 0, -\alpha_4 + \alpha_5$. Furthermore, if $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5 \neq 0$, it follows that $c_{0,0} = 1/2,\ \alpha_5/(-\alpha_4 + \alpha_5)$.*
(1) *Assume that $b_{0,0} + d_{0,0} = 0$.*
 (i) *If $b_{0,0} = 0$,*

$$h_{0,0} = 0.$$

 (ii) *If $b_{0,0} \neq 0$,*

$$h_{0,0} = -\alpha_2(\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5).$$

(2) *Assume that $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5 \neq 0$ and $c_{0,0} = 1/2$.*

(i)-1 *If* $2\alpha_3+\alpha_4+\alpha_5=0$ *and* $b_{0,0}=0$, *then*

$$h_{0,0}=\alpha_4^2.$$

(i)-2 *If* $2\alpha_3+\alpha_4+\alpha_5=0$ *and* $b_{0,0}\neq 0$, *then*

$$h_{0,0}=-\alpha_2^2+\alpha_5^2.$$

(ii) *If* $2\alpha_3+\alpha_4+\alpha_5\neq 0$,

$$h_{0,0}=-\alpha_2(\alpha_2+2\alpha_3+\alpha_4+\alpha_5)-(\alpha_3+\alpha_4)(\alpha_3+\alpha_5).$$

(3) *Assume that* $b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5\neq 0$ *and* $c_{0,0}=\alpha_5/(-\alpha_4+\alpha_5)\neq 1/2$.
(i) *If* $b_{0,0}=0$,

$$h_{0,0}=-\alpha_4\alpha_5.$$

(ii) *If* $b_0\neq 0$,

$$h_{0,0}=-\alpha_2(\alpha_2+2\alpha_3+\alpha_4+\alpha_5)-\alpha_4\alpha_5.$$

4.3.2 The case where x has a pole at $t=0$

Proposition 4.11. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, *there exists a solution such that* x *has a pole at* $t=0$ *and* y,z,w *are all holomorphic at* $t=0$. *Then,* $d_{0,0}=0,-\alpha_4+\alpha_5$.
(1) *If case* (1) *occurs in Proposition 2.8, or if* $d_{0,0}=0$, *then*

$$h_{0,0}=\alpha_1(1-\alpha_0).$$

(2) *If case* (2) *occurs in Proposition 2.8, or if* $d_{0,0}=-\alpha_4+\alpha_5\neq 0$, *then*

$$h_{0,0}=\alpha_1(1-\alpha_0)-\alpha_4\alpha_5.$$

4.3.3 The case where z has a pole at $t=0$

Proposition 4.12. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, *there exists a solution such that* z *has a pole of order* n $(n\geq 1)$ *at* $t=0$ *and* x,y,w *are holomorphic at* $t=0$. *Then,*

$$h_{0,0}=\alpha_3(\alpha_3+\alpha_4+\alpha_5).$$

4.3.4 The case where x, z have a pole at $t = 0$

Proposition 4.13. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that x has a pole of order one at $t = 0$ and z has a pole of order n ($n \ge 1$) at $t = 0$ and y, w are both holomorphic at $t = 0$.*
(1) *Assume that $n \ge 2$. Then,*

$$h_{0,0} = \frac{1}{4}(-\alpha_0 + \alpha_1)^2 + \frac{1}{2}(-\alpha_0 + \alpha_1) + \alpha_3(\alpha_3 + \alpha_4 + \alpha_5).$$

(2) *Assume that $n = 1$. Then, $b_{0,1} = -1/2,\ \alpha_1/(\alpha_0 - \alpha_1)$.*
(i) *If $b_{0,1} = -1/2$,*

$$h_{0,0} = \frac{1}{4}(-\alpha_0 + \alpha_1 + 1)^2 - \frac{1}{4}(\alpha_4 + \alpha_5)^2.$$

(ii) *If $b_{0,1} = \alpha_1/(\alpha_0 - \alpha_1) \ne -1/2$,*

$$h_{0,0} = \alpha_1(1 - \alpha_0) + \alpha_3(\alpha_3 + \alpha_4 + \alpha_5).$$

4.3.5 The case where y, w have a pole at $t = 0$

Proposition 4.14. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that y, w both have a pole of order n ($n \ge 1$) at $t = 0$ and x, z are both holomorphic at $t = 0$. Then,*

$$h_{0,0} = \frac{1}{4}(-\alpha_4 + \alpha_5)^2 - \alpha_2(\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5).$$

4.4 The Laurent series of H at $t = c \in \mathbb{C}^*$

In this subsection, we compute the residue $h_{c,-1}$ of H at $t = c \in \mathbb{C}^*$ for a meromorphic solution of $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ at $t = c$.

4.4.1 The case where x has a pole at $t = c \in \mathbb{C}^*$

Proposition 4.15. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that x has a pole of order one at $t = c \in \mathbb{C}^*$ and y, z, w are all holomorphic at $t = c$. Then, $b_{c,0} = 0, -c$.*
(1) *If $b_{c,0} = 0$, H is holomorphic at $t = c$.*
(2) *If $b_{c,0} = -c$, H has a pole of order one at $t = c$ and*

$$h_{c,-1} = Res_{t=c} H = c.$$

4.4.2 The case where y has a pole at $t = c \in \mathbb{C}^*$

Proposition 4.16. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution such that y has a pole of order one at $t = c \in \mathbb{C}^*$ and x, z, w are all holomorphic at $t = c$. H is then holomorphic at $t = c$.*

4.4.3 The case where z has a pole at $t = c \in \mathbb{C}^*$

Proposition 4.17. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution such that z has a pole of order n $(n \ge 1)$ at $t = c \in \mathbb{C}^*$ and x, y, w are all holomorphic at $t = c \in \mathbb{C}^*$. H is then holomorphic at $t = c$.*

4.4.4 The case where w has a pole at $t = c \in \mathbb{C}^*$

Proposition 4.18. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution such that w has a pole of order one at $t = c \in \mathbb{C}^*$ and x, y, z are all holomorphic at $t = c$. H is then holomorphic at $t = c$.*

4.4.5 The case where x, z have a pole at $t = c \in \mathbb{C}^*$

Proposition 4.19. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a solution such that x, z both have a pole of order one at $t = c \in \mathbb{C}^*$ and y, w are both holomorphic at $t = c$. Then, $(b_{c,0}, d_{c,0}) = (0, 0), (0, -c), (-c, 0), (-c, c)$.*

(1) *If $(b_{c,0}, d_{c,0}) = (0, 0)$, H is holomorphic at $t = c$.*

(2) *Assume that $(b_{c,0}, d_{c,0}) = (0, -c)$. Then, $(a_{c,-1}, c_{c,-1}) = (-2, 1),\ (1, 1)$.*

(i) *If $(a_{c,-1}, c_{c,-1}) = (-2, 1)$, H has a pole of order one at $t = c$ and*

$$h_{c,-1} = Res_{t=c} H = c.$$

(ii) *If $(a_{c,-1}, c_{c,-1}) = (1, 1)$, H has a pole of order one at $t = c$ and*

$$h_{c,-1} = Res_{t=c} H = c.$$

(3) *If $(b_{c,0}, d_{c,0}) = (-c, 0)$, H has a pole of order one at $t = c$ and*

$$h_{c,-1} = Res_{t=c} H = c.$$

(4) *Assume that $(b_{c,0}, d_{c,0}) = (-c, c)$. Then, $(a_{c,-1}, c_{c,-1}) = (2, -1),\ (-1, -1)$.*

(i) *If $(a_{c,-1}, c_{c,-1}) = (2, -1)$, H has a pole of order one at $t = c$ and*

$$h_{c,-1} = Res_{t=c} H = 3c.$$

(ii) *If $(a_{c,-1}, c_{c,-1}) = (-1, -1)$, H is holomorphic at $t = c$.*

4.4.6 The case where x, w have a pole at $t = c \in \mathbb{C}^*$

Proposition 4.20. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a solution such that x, w both have a pole of order one at $t = c \in \mathbb{C}^*$ and y, z are both holomorphic at $t = c$. Then, $(b_{c,0}, c_{c,0}) = (0,0), (0,1), (-c,0), (-c,1)$.*
(1) *If $(b_{c,0}, c_{c,0}) = (0,0)$, H is holomorphic at $t = c$.*
(2) *If $(b_{c,0}, c_{c,0}) = (0,1)$, H is holomorphic at $t = c$.*
(3) *If $(b_{c,0}, c_{c,0}) = (-c,0)$, H has a pole of order one at $t = c$ and*

$$h_{c,-1} = Res_{t=c}H = c.$$

(4) *If $(b_{c,0}, c_{c,0}) = (-c,1)$, H has a pole of order one at $t = c$ and*

$$h_{c,-1} = Res_{t=c}H = c.$$

4.4.7 The case where y, w have a pole at $t = c \in \mathbb{C}^*$

Proposition 4.21. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exist a solution such that y, w both have a pole of order n $(n \ge 1)$ at $t = c \in \mathbb{C}^*$ and x, z are both holomorphic at $t = c$. H is then holomorphic at $t = c$.*

4.5 A rational solution and its Hamiltonian H

In this subsection, we obtain relation between $h_{\infty,0}$ and $h_{0,0}$ for a rational solution of $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$.

Proposition 4.22. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution. Then,*

$$h_{\infty,0} - h_{0,0} = m,$$

where m is a non-negative integer.

Proof. From the discussions in the previous subsection, H has a pole of at most order one at $t = c \in \mathbb{C}^*$ and the residue of H at $t = c \in \mathbb{C}^*$ is expressed by $n \times c$, where n is a non-negative integer. Then, if some of x, y, z, w have a pole at $t = c_1, c_2, \ldots, c_k \in \mathbb{C}^*$, it follows that

$$\begin{aligned} H = {} & h_{\infty,n_\infty}t^{n_\infty} + h_{\infty,n_\infty-1}t^{n_\infty-1} + \cdots + h_{\infty,0} \\ & + h_{0,-n_0}t^{-n_0} + h_{0,-(n_0-1)}t^{-(n_0-1)} + \cdots + h_{0,-1}t^{-1} \\ & + \sum_{l=1}^{k}\frac{n_l c_l}{t-c_l}, \end{aligned}$$

where n_∞, n_0, n_l are non-negative integers. Thus, by comparing the constant terms of H at $t=0$, we find that

$$h_{\infty,0} - \sum_{l=1}^{k} n_l = h_{0,0},$$

which proves the proposition. □

5 The properties of the Bäcklund transformations

In this section, we investigate the properties of the Bäcklund transformations.

5.1 The properties of s_0, s_1, s_2, s_3, s_4 and s_5

Proposition 5.1. (0) *If $y+t \equiv 0$ for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, then $\alpha_0 = 0$.*
(1) *If $y \equiv 0$ for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, then $\alpha_1 = 0$.*
(2) *If $x - z \equiv 0$ for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, then $\alpha_2 = 0$ or $x = z \equiv 1/2$.*
(3) *If $w \equiv 0$ for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, then $\alpha_3 = 0$ or $y = -t/2$.*
(4) *If $z - 1 \equiv 0$ for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, then $\alpha_4 = 0$.*
(5) *If $z \equiv 0$ for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, then $\alpha_4 = 0$.*

Proof. We show case (2). The other cases can be proved in the same way. Considering

$$\begin{cases} tx' = 2x^2y + tx^2 - 2xy - \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\}\, x \\ \qquad\qquad\qquad\qquad + (\alpha_2 + \alpha_5) + 2z\,\{(z-1)w + \alpha_3\}, \\ tz' = 2z^2w + tz^2 - 2zw - \{t + (\alpha_5 + \alpha_4)\}\, z + \alpha_5 + 2yz(z-1), \end{cases}$$

we have

$$\alpha_2(1 - 2x) = 0,$$

which implies that

$$\alpha_2 = 0 \text{ or } x = z \equiv \frac{1}{2}.$$

□

By Proposition 5.1, we consider s_0, s_1, s_4 or s_5 as the identical transformation if case (0), (1), (4) or (5) occurs in Proposition 5.1. Moreover, we consider s_2 or s_3 as the identical transformation if $x - z \equiv 0$ and $\alpha_2 = 0$, or if $w \equiv 0$ and $\alpha_3 = 0$.

5.2 More on the properties of s_2

Proposition 5.2. *Suppose that $x = z \equiv 1/2$ and $\alpha_2 \neq 0$. One of the following then occurs.*

(1) $\alpha_0 = \alpha_1 = 1/2$ *and*

$$y = -\frac{1}{2}t - \frac{1}{2\alpha_2}(\alpha_4 + \alpha_5)(-\alpha_4 + \alpha_5),\ w = (-\alpha_4 + \alpha_5) + \frac{1}{2\alpha_2}(\alpha_4 + \alpha_5)(-\alpha_4 + \alpha_5).$$

(2) $-\alpha_0 + \alpha_1 = 0,\ (\alpha_4 + \alpha_5)(-\alpha_4 + \alpha_5) = 0$ *and*

$$y = -\frac{1}{2}t,\ w = -\alpha_4 + \alpha_5.$$

(3) $\alpha_0 + \alpha_1 + 2\alpha_2 = 0,\ (-\alpha_4 + \alpha_5)(\alpha_4 + \alpha_5) = 0$ *and*

$$y = -\frac{1}{2}t + \frac{-\alpha_0 + \alpha_1}{4\alpha_2}t,\ w = -\frac{-\alpha_0 + \alpha_1}{4\alpha_2}t + (-\alpha_4 + \alpha_5).$$

(4) $\alpha_0 + \alpha_1 = 1,\ \alpha_2 = -1/2$ *and*

$$y = -\frac{1}{2}t - \frac{-\alpha_0 + \alpha_1}{2}t + (\alpha_4 + \alpha_5)(-\alpha_4 + \alpha_5),\ w = \frac{-\alpha_0 + \alpha_1}{2}t - (\alpha_4 + \alpha_5)(-\alpha_4 + \alpha_5) + (-\alpha_4 + \alpha_5).$$

Proof. It can be proved by direct calculations. □

Proposition 5.2 implies that s_2 transforms the solutions in Proposition 5.2 into a solution such that $y = w \equiv \infty$, which is called the "infinite solution."

5.3 More on the properties of s_3

Proposition 5.3. *Suppose that $w \equiv 0$ and $\alpha_3 \neq 0$. One of the following then occurs.*

(1) $(\alpha_0 + \alpha_1)(-\alpha_0 + \alpha_1) = 0$ *and* $-\alpha_4 + \alpha_5 = 0$ *and*

$$x = \frac{1}{2} + (-\alpha_0 + \alpha_1)t^{-1},\ y = -\frac{1}{2}t,\ z = \frac{1}{2},\ w = 0,$$

(2) $(\alpha_0 + \alpha_1)(-\alpha_0 + \alpha_1) = 0$ *and* $2\alpha_3 + \alpha_4 + \alpha_5 = 0$ *and*

$$x = \frac{1}{2} + (-\alpha_0 + \alpha_1)t^{-1},\ y = -\frac{1}{2}t,\ z = \frac{1}{2} - \frac{-\alpha_4 + \alpha_5}{4\alpha_3},\ w = 0,$$

(3) $\alpha_4 = \alpha_5 = 1/2$ *and*

$$x = \frac{1}{2} + (-\alpha_0 + \alpha_1)t^{-1},\ y = -\frac{1}{2}t,\ z = \frac{1}{2} - \frac{(\alpha_0 + \alpha_1)(-\alpha_0 + \alpha_1)}{2\alpha_3}t^{-1},\ w = 0,$$

(4) $\alpha_4 + \alpha_5 = 1$ *and* $\alpha_3 = -1/2$ *and*

$$x = \frac{1}{2} + (-\alpha_0 + \alpha_1)t^{-1},\ y = -\frac{1}{2}t,\ z = \frac{1}{2} + \frac{-\alpha_4 + \alpha_5}{2} + (\alpha_0 + \alpha_1)(-\alpha_0 + \alpha_1)t^{-1},\ w = 0.$$

Proof. It can be proved by direct calculations. $\square$

Proposition 5.3 implies that s_3 transforms the solutions in Proposition 5.3 into a solution such that $z \equiv \infty$, which is called the "infinite solution."

6 Infinite solutions

In the previous section, we have suggested a solution such that $y = w \equiv \infty$ or $z \equiv \infty$, which is called a "infinite solution." In this section, we treat such a solution. For this purpose, following Sasano [32], we introduce the coordinate transformations, which are defined by

$$m_2 : x_2 = -\{(x-z)y - \alpha_2\}y, \ y_2 = \frac{1}{y}, \ z_2 = z, \ w_2 = w + y, \tag{6.1}$$

$$m_3 : x_3 = x, \ y_3 = y, \ z_3 = \frac{1}{z}, \ w_3 = -z(wz + \alpha_3), \tag{6.2}$$

We note that the case in which $y_2 \equiv 0$ corresponds to the case in which $y \equiv w \equiv \infty$ and $x, z \not\equiv \infty$, and the case in which $z_3 \equiv 0$ corresponds to the case in which $z \equiv \infty$ and $x, y, w \not\equiv \infty$.

6.1 The case in which $y, w \equiv \infty$

By setting $y_2 \equiv 0$, we can prove the following proposition:

Proposition 6.1. $D_5^{(1)}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ *has a rational solution such that* $y, w \equiv \infty$, *if and only if one of the following occurs:*
(1) $(\alpha_0 + \alpha_1)(-\alpha_0 + \alpha_1) = 0$ *and* $\alpha_0 + \alpha_1 + 2\alpha_2 = 1$,
(2) $(\alpha_0 + \alpha_1)(-\alpha_0 + \alpha_1) = 0$ *and* $(\alpha_4 + \alpha_5)(-\alpha_4 + \alpha_5) = 0$.
Furthermore,

$$x_2 = -\frac{1}{4}(\alpha_0 - \alpha_1 - 2\alpha_2)t - \frac{1}{2}(-\alpha_4 + \alpha_5)(\alpha_4 + \alpha_5), \ y_2 = 0, \ z_2 = \frac{1}{2}, \ w_2 = -\frac{1}{2}t + (-\alpha_4 + \alpha_5),$$

or

$$x = \frac{1}{2}, \ y = \infty, \ z = \frac{1}{2}, \ w = \infty.$$

Proof. We first note that m_2 transforms the system of (x, y, z, w) into the system of (x_2, y_2, z_2, w_2), which is given by

$$\begin{cases} tx_2' = 2x_2^2 y_2 + 3tx_2^2 y_2^2 - 2tx_2 z_2 + \{t + (2\alpha_3 + \alpha_4 + \alpha_5)\}x_2 - 2(\alpha_1 + 2\alpha_2)tx_2 y_2 + \alpha_2(\alpha_1 + \alpha_2)t, \\ ty_2' = -2x_2 y_2^2 + 2z_2 - 1 - 2tx_2 y_2^3 + 2ty_2 z_2 - \{t + (2\alpha_3 + \alpha_4 + \alpha_5)\}y_2 + t(\alpha_1 + 2\alpha_2)y_2^2, \\ tz_2' = 2z_2^2 w_2 + tz_2^2 - 2z_2 w_2 - \{t + (\alpha_5 + \alpha_4)\}z_2 + \alpha_5, \\ tw_2' = -2z_2 w_2^2 + w_2^2 - 2tz_2 w_2 + \{t + (\alpha_5 + \alpha_4)\}w_2 + 2x_2 + 2tx_2 y_2 - (\alpha_1 + 2\alpha_2 + \alpha_3)t. \end{cases}$$

Substituting $y_2 \equiv 0$ in

$$ty_2' = -2x_2y_2^2 + 2z_2 - 1 - 2tx_2y_2^3 + 2ty_2z_2 - \{t + (2\alpha_3 + \alpha_4 + \alpha_5)\}y_2 + t(\alpha_1 + 2\alpha_2)y_2^2,$$

we have $z_2 \equiv 1/2$.

Considering

$$tz_2' = 2z_2^2w_2 + tz_2^2 - 2z_2w_2 - \{t + (\alpha_5 + \alpha_4)\}z_2 + \alpha_5,$$

we then obtain $w_2 = -t/2 + (-\alpha_4 + \alpha_5)$.

Therefore, considering

$$tw_2' = -2z_2w_2^2 + w_2^2 - 2tz_2w_2 + \{t + (\alpha_5 + \alpha_4)\}w_2 + 2x_2 + 2tx_2y_2 - (\alpha_1 + 2\alpha_2 + \alpha_3)t,$$

we have $x_2 = -(\alpha_0 - \alpha_1 - 2\alpha_2)t/4 - (-\alpha_4 + \alpha_5)(\alpha_4 + \alpha_5)/2$.

Thus, substituting the solution in

$$tx_2' = 2x_2^2y_2 + 3tx_2^2y_2^2 - 2tx_2z_2 + \{t + (2\alpha_3 + \alpha_4 + \alpha_5)\}x_2 - 2(\alpha_1 + 2\alpha_2)tx_2y_2 + \alpha_2(\alpha_1 + \alpha_2)t,$$

we obtain

$$\frac{1}{4}(\alpha_0 - \alpha_1)(\alpha_0 + \alpha_1)t + \frac{1}{2}(\alpha_0 + \alpha_1 + 2\alpha_2 - 1)(-\alpha_4 + \alpha_5)(\alpha_4 + \alpha_5) = 0,$$

which proves the proposition. □

6.2 The case in which $z \equiv \infty$

By setting $z_3 \equiv 0$, we can show the following proposition:

Proposition 6.2. *$D_5^{(1)}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ has a rational solution such that $z \equiv \infty$, if and only if one of the following occurs:*
(1) $(\alpha_0 + \alpha_1)(-\alpha_0 + \alpha_1) = 0$ *and* $(\alpha_4 + \alpha_5)(-\alpha_4 + \alpha_5) = 0$,
(2) $2\alpha_3 + \alpha_4 + \alpha_5 = 1$ *and* $(\alpha_4 + \alpha_5)(-\alpha_4 + \alpha_5) = 0$.
Furthermore,

$$x_3 = \frac{1}{2} + (-\alpha_0 + \alpha_1)t^{-1},\ y_3 = -\frac{1}{2}t,\ z_3 = 0,\ w_3 = \frac{1}{4}(2\alpha_3 - \alpha_4 + \alpha_5) + \frac{1}{2}(-\alpha_0 + \alpha_1)(\alpha_0 + \alpha_1)t^{-1},$$

or

$$x = \frac{1}{2} + (-\alpha_0 + \alpha_1)t^{-1},\ y = -\frac{1}{2}t,\ z = \infty,\ w = 0.$$

Proof. We first note that m_3 transforms the system of (x, y, z, w) into the system of (x_3, y_3, z_3, w_3), which is given by

$$\begin{cases} tx_3' = 2x_3^2y_3 + tx_3^2 - 2x_3y_3 - \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\}x_3 + \alpha_2 + \alpha_5 - 2w_3 + 2z_3w_3 + 2\alpha_3, \\ ty_3' = -2x_3y_3^2 + y_3^2 - 2tx_3y_3 + \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\}y_3 - \alpha_1 t, \\ tz_3' = 2z_3^2 - 2z_3^3w_3 + \{t + (2\alpha_3 + \alpha_4 + \alpha_5)\}z_3 - t - (2\alpha_3 + \alpha_5)z_3^2 - 2y_3 + 2y_3z_3, \\ tw_3' = -2z_3w_3^2 - \{t + (2\alpha_3 + \alpha_4 + \alpha_5)\}w_3 + 3z_3^2w_3^2 + 2(2\alpha_3 + \alpha_5)z_3w_3 - 2y_3w_3 + \alpha_3(\alpha_3 + \alpha_5). \end{cases}$$

Substituting $z_3 \equiv 0$ in

$$tz_3' = 2z_3^2 - 2z_3^3w_3 + \{t + (2\alpha_3 + \alpha_4 + \alpha_5)\}z_3 - t - (2\alpha_3 + \alpha_5)z_3^2 - 2y_3 + 2y_3z_3,$$

we have $y_3 = -t/2$.

Considering

$$ty_3' = -2x_3y_3^2 + y_3^2 - 2tx_3y_3 + \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\}y_3 - \alpha_1 t,$$

we then obtain $x_3 = 1/2 + (-\alpha_0 + \alpha_1)t^{-1}$.

Therefore, considering

$$tx_3' = 2x_3^2y_3 + tx_3^2 - 2x_3y_3 - \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\}x_3 + \alpha_2 + \alpha_5 - 2w_3 + 2z_3w_3 + 2\alpha_3,$$

we have

$$w_3 = \frac{1}{2}(-\alpha_0 + \alpha_1)(\alpha_0 + \alpha_1)t^{-1} + \frac{1}{4}(2\alpha_3 - \alpha_4 + \alpha_5).$$

Thus, substituting the solution in

$$tw_3' = -2z_3w_3^2 - \{t + (2\alpha_3 + \alpha_4 + \alpha_5)\}w_3 + 3z_3^2w_3^2 + 2(2\alpha_3 + \alpha_5)z_3w_3 - 2y_3w_3 + \alpha_3(\alpha_3 + \alpha_5),$$

we obtain

$$-\frac{1}{2}(-\alpha_0 + \alpha_1)(\alpha_0 + \alpha_1)(2\alpha_3 + \alpha_4 + \alpha_5 - 1)t^{-1} - \frac{1}{4}(-\alpha_4 + \alpha_5)(\alpha_4 + \alpha_5) = 0,$$

which proves the proposition. □

6.3 Bäcklund transformations and infinite solutions

If $x = z \equiv 1/2$ and $\alpha_2 \neq 0$, or if $w \equiv 0$ and $\alpha_3 \neq 0$, s_2 or s_3 is defined by

$$(x, y, z, w) \longrightarrow \left(\frac{1}{2}, \infty, \frac{1}{2}, \infty\right),$$
$$(x, y, z, w) \longrightarrow \left(\frac{1}{2} + (-\alpha_0 + \alpha_1)t^{-1}, -\frac{1}{2}t, \infty, 0\right),$$

respectively.

From now on, we consider relationship between Bäcklund transformations and infinite solutions.

6.3.1 The case where $y \equiv w \equiv \infty$

By Proposition 6.1 and m_2, we can prove the following:

Proposition 6.3. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, $y \equiv w \equiv \infty$. The action of the Bäcklund transformations are then expressed as follows:*
(0) $s_0 : (1/2, \infty, 1/2, \infty) \longrightarrow (1/2, \infty, 1/2, \infty)$,
(1) $s_1 : (1/2, \infty, 1/2, \infty) \longrightarrow (1/2, \infty, 1/2, \infty)$,
(2) *if* $\alpha_2 = 0$, $s_2 : (1/2, \infty, 1/2, \infty) \longrightarrow (1/2, \infty, 1/2, \infty)$,
if $\alpha_2 \neq 0$, *by* s_2, *the solution is transformed so that*

$$x = \frac{1}{2},\ y = -\frac{1}{2}t - \frac{-\alpha_0 + \alpha_1}{4\alpha_2}t + \frac{(-\alpha_4 + \alpha_5)(\alpha_4 + \alpha_5)}{2\alpha_2},\ z = \frac{1}{2},\ w = \frac{-\alpha_0 + \alpha_1}{4\alpha_2}t - \frac{(-\alpha_4 + \alpha_5)(\alpha_4 + \alpha_5)}{2\alpha_2},$$

(3) $s_3 : (1/2, \infty, 1/2, \infty) \longrightarrow (1/2, \infty, 1/2, \infty)$,
(4) $s_4 : (1/2, \infty, 1/2, \infty) \longrightarrow (1/2, \infty, 1/2, \infty)$,
(5) $s_5 : (1/2, \infty, 1/2, \infty) \longrightarrow (1/2, \infty, 1/2, \infty)$,
(6) $\pi_1 : (1/2, \infty, 1/2, \infty) \longrightarrow (1/2, \infty, 1/2, \infty)$,
(7) *by* π_2, *the solution is transformed so that*

$$x = \frac{1}{2} - (-\alpha_4 + \alpha_5)t^{-1},\ y = -\frac{1}{2}t,\ z = \infty,\ w = 0.$$

(8) $\pi_3 : (1/2, \infty, 1/2, \infty) \longrightarrow (1/2, \infty, 1/2, \infty)$,
(9) $\pi_4 : (1/2, \infty, 1/2, \infty) \longrightarrow (1/2, \infty, 1/2, \infty)$.

Proof. By Proposition 6.1, we first note that

$$x_2 = -\frac{1}{4}(\alpha_0 - \alpha_1 - 2\alpha_2)t - \frac{1}{2}(-\alpha_4 + \alpha_5)(\alpha_4 + \alpha_5),\ y_2 = 0,\ z_2 = \frac{1}{2},\ w_2 = -\frac{1}{2}t + (-\alpha_4 + \alpha_5).$$

We treat case (2), especially the case where $\alpha_2 \neq 0$. The other cases can be proved in the same way.

From the definition of s_2 and m_2, it follows that

$$\begin{aligned}
s_2 m_2^{-1}(x_2) \quad &= (-x_2 y_2 + \alpha_2)y_2 + z_2 = 1/2,\\
s_2 m_2^{-1}(y_2) \quad &= -x_2/(-x_2 y_2 + \alpha_2)\\
&= -t/2 - (-\alpha_0 + \alpha_1)t/4\alpha_2 + (-\alpha_4 + \alpha_5)(\alpha_4 + \alpha_5)/2\alpha_2,\\
s_2 m_2^{-1}(z_2) \quad &= z_2 = 1/2,\\
s_2 m_2^{-1}(w_2) \quad &= w_2 + x_2/(-x_2 y_2 + \alpha_2)\\
&= (-\alpha_0 + \alpha_1)t/\{4\alpha_2\} + (-\alpha_4 + \alpha_5) - (-\alpha_4 + \alpha_5)(\alpha_4 + \alpha_5)/\{2\alpha_2\}.
\end{aligned}$$

$\square$

6.3.2 The case where $z \equiv \infty$

Proposition 6.4. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, $z \equiv \infty$. The action of the Bäcklund transformations are then expressed as follows:*
(0) $s_0 : (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0) \longrightarrow (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0)$,
(1) $s_1 : (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0) \longrightarrow (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0)$,
(2) $s_2 : (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0) \longrightarrow (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0)$,
(3) *if* $\alpha_3 = 0$, $s_3 : (1/2, \infty, 1/2, \infty) \longrightarrow (1/2, \infty, 1/2, \infty)$,
if $\alpha_3 \neq 0$, *by* s_3, *the solution is transformed so that*

$$x = \frac{1}{2} + (-\alpha_0 + \alpha_1)t^{-1},\ y = -\frac{1}{2}t,\ z = \frac{1}{2} + \frac{-\alpha_4 + \alpha_5}{4\alpha_3} + \frac{(-\alpha_0 + \alpha_1)(\alpha_0 + \alpha_1)}{2\alpha_3}t^{-1},\ w = 0.$$

(4) $s_4 : (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0) \longrightarrow (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0)$,
(5) $s_5 : (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0) \longrightarrow (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0)$,
(6) $\pi_1 : (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0) \longrightarrow (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0)$,
(7) *by* π_2, *the solution is transformed so that*

$$x = \frac{1}{2},\ y = \infty,\ z = \frac{1}{2},\ w = \infty,$$

(8) $\pi_3 : (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0) \longrightarrow (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0)$,
(9) $\pi_4 : (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0) \longrightarrow (1/2 + (-\alpha_0 + \alpha_1)t^{-1}, -t/2, \infty, 0)$.

Proof. By Proposition 6.2, we first note that

$$x_3 = \frac{1}{2}+(-\alpha_0+\alpha_1)t^{-1},\ y_3 = -\frac{1}{2}t,\ z_3 = 0,\ w_3 = \frac{1}{4}(2\alpha_3-\alpha_4+\alpha_5)+\frac{1}{2}(-\alpha_0+\alpha_1)(\alpha_0+\alpha_1)t^{-1},$$

We treat case (3), especially the case where $\alpha_3 \neq 0$. The other cases can be proved in the same way.

From the definition of s_3 and m_3, it follows that

$$\begin{aligned}
s_3m_3^{-1}(x_3) &= x_3 = 1/2 + (-\alpha_0 + \alpha_1)t^{-1},\\
s_3m_3^{-1}(y_3) &= y_3 = -t/2,\\
s_3m_3^{-1}(z_3) &= w_3/(z_3w_3 + \alpha_3) = 1/2 + (-\alpha_4 + \alpha_5)/\{4\alpha_3\} + (-\alpha_0 + \alpha_1)(\alpha_0 + \alpha_1)/\{2\alpha_3 t\},\\
s_2m_2^{-1}(w_2) &= -z_3(z_3w_3 + \alpha_3) = 0.
\end{aligned}$$

□

7 Necessary conditions for type A

In this section, we obtain necessary conditions for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ to have rational solutions of type A.

7.1 The case in which x, y, z, w are all holomorphic at $t = \infty$

7.1.1 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$

Let us treat the case where x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = 0$ and $b_{0,0} = 0$.

Proposition 7.1. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type A such that x, y, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. Moreover, assume that x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = 0$ and $b_{0,0} = 0$. One of the following then occurs:*
(1) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$, (2) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z}$.

Proof. Since $a_{\infty,-1} \in \mathbb{Z}$ and $(b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}$, it follows from Proposition 1.1 that $\alpha_2 + \alpha_5 \in \mathbb{Z}$ and $\alpha_1 + \alpha_3 \in \mathbb{Z}$. In order to obtain more necessary conditions, we consider the case where $\alpha_1 \neq 0$ and the case where $\alpha_4 \neq 0$.

We prove that $\alpha_1 \in \mathbb{Z}$, if $\alpha_1 \neq 0$. Since $\alpha_1 \neq 0$, $s_1(x, y, z, w)$ is a rational solution of $D_5^{(1)}(\alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4, \alpha_5)$ such that all of $s_1(x, y, z, w)$ are holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$. Moreover, $s_1(x)$ has a pole at $t = 0$ and all of $s_1(y, z, w)$ are holomorphic at $t = 0$. Thus, we can obtain $\alpha_1 \in \mathbb{Z}$, because $\alpha_2+\alpha_5 \in \mathbb{Z}$ and $a_{\infty,-1}-a_{0,-1} \in \mathbb{Z}$ for $s_1(x, y, z, w)$.

We prove that $\alpha_5 \in \mathbb{Z}$, if $\alpha_4 \neq 0$. Since $\alpha_4 \neq 0$, $s_4(x, y, z, w)$ is a rational solution of $D_5^{(1)}(\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4, \alpha_5)$ such that all of $s_1(x, y, z, w)$ are holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. Moreover, all of $s_4(x, y, z, w)$ are holomorphic at $t = 0$ and for $s_4(x, y, z, w)$, $b_{0,0} + d_{0,0} = -(-\alpha_4) + \alpha_5$. Thus, we can obtain $\alpha_5 \in \mathbb{Z}$, because for $s_4(x, y, z, w)$, $(b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}$.

Now, let us consider the following four cases:
(1) $\alpha_1 \neq 0,\ \alpha_4 \neq 0$, (2) $\alpha_1 \neq 0,\ \alpha_4 = 0$, (3) $\alpha_1 = 0,\ \alpha_4 \neq 0$, (4) $\alpha_1 = \alpha_4 = 0$.

If case (1) or (3) occurs, then it follows that $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$. If case (2) or (4) occurs, it then follows that $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z}$. □

Let us deal with the case where x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0}+d_{0,0} = 0$ and $b_{0,0} \neq 0$. By s_1 and s_5, we can then prove the following proposition in the same way:

Proposition 7.2. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type A such that x, y, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. Moreover, assume that x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = 0$ and $b_{0,0} \neq 0$. One of the following then occurs:*
(1) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$, (2) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z}$.

Proof. Considering $a_{\infty,-1} \in \mathbb{Z}$ and $(b_{\infty,0}+d_{\infty,0})-(b_{0,0}+d_{0,0}) \in \mathbb{Z}$, we find that $\alpha_2+\alpha_5 \in \mathbb{Z}$ and $\alpha_1 + \alpha_3 \in \mathbb{Z}$.

We first prove that if $\alpha_1 \neq 0$, $\alpha_2+\alpha_3+\alpha_4 \in \mathbb{Z}$. Since $\alpha_1 \neq 0$, it follows that $s_1(x,y,z,w)$ is a rational solution of $D_5^{(1)}(\alpha_0, -\alpha_1, \alpha_2+\alpha_1, \alpha_3, \alpha_4, \alpha_5)$ such that all of $s_1(x,y,z,w)$ are holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1,0)$. Moreover, all of $s_1(x,y,z,w)$ are holomorphic at $t=0$. We then obtain $\alpha_2+\alpha_3+\alpha_4 \in \mathbb{Z}$ because $a_{\infty,-1} \in \mathbb{Z}$ for $s_1(x,y,z,w)$.

We next show that if $\alpha_5 \neq 0$, $\alpha_0+\alpha_3+\alpha_4 \in \mathbb{Z}$. Since $\alpha_5 \neq 0$, it follows that $s_5(w)$ has a pole at $t=\infty$ and all $s_5(x,y,z)$ are holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0,0)$. Moreover, all of $s_5(x,y,z,w)$ are holomorphic at $t=0$ and for $s_5(x,y,z,w)$, $b_{0,0}+d_{0,0} = -\alpha_4-\alpha_5$. We then obtain $\alpha_0+\alpha_3+\alpha_4 \in \mathbb{Z}$ because for $s_5(x,y,z,w)$, $(b_{\infty,0}+d_{\infty,0}) - (b_{0,0}+d_{0,0}) \in \mathbb{Z}$.

Therefore, if $\alpha_1 \neq 0$, we have $\alpha_0 \in \mathbb{Z}, \alpha_2 \in \mathbb{Z}, \alpha_5 \in \mathbb{Z}, \alpha_3+\alpha_4 \in \mathbb{Z}$. If $\alpha_1 = 0$, we obtain $\alpha_1 \in \mathbb{Z}, \alpha_2 \in \mathbb{Z}, \alpha_3 \in \mathbb{Z}, \alpha_5 \in \mathbb{Z}$. ☐

Let us treat the case where x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}+d_{0,0} = -\alpha_4+\alpha_5 \neq 0$ and $b_{0,0} = 0$.

Proposition 7.3. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a rational solution of type A such that x,y,z,w are all holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0,0)$. Moreover, assume that x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}+d_{0,0} = -\alpha_4+\alpha_5 \neq 0$ and $b_{0,0} = 0$. Then,*

$$\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}.$$

Proof. Since $a_{\infty,-1} \in \mathbb{Z}$ and $(b_{\infty,0}+d_{\infty,0}) - (b_{0,0}+d_{0,0}) \in \mathbb{Z}$, it follows from Proposition 1.1 that $\alpha_2+\alpha_5 \in \mathbb{Z}$ and $\alpha_1+\alpha_3+\alpha_4-\alpha_5 \in \mathbb{Z}$.

We first prove that if $\alpha_1 \neq 0$, $\alpha_0+\alpha_1+\alpha_3 \in \mathbb{Z}$. Since $\alpha_1 \neq 0$, $s_1(x,y,z,w)$ is a rational solution of $D_5^{(1)}(\alpha_0, -\alpha_1, \alpha_2+\alpha_1, \alpha_3, \alpha_4, \alpha_5)$ such that all of $s_1(x,y,z,w)$ are holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1,0)$. Moreover, $s_1(x)$ has a pole at $t=0$ and all of $s_1(y,z,w)$ are holomorphic at $t=0$. We then obtain $\alpha_0+\alpha_1+\alpha_3 \in \mathbb{Z}$ because for $s_1(x,y,z,w)$, $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$.

Considering $(b_{\infty,0}+d_{\infty,0}) - (b_{0,0}+d_{0,0}) \in \mathbb{Z}$ for $s_4(x,y,z,w)$, we obtain

$$\begin{cases} \alpha_4 \in \mathbb{Z} \text{ if } \alpha_4 \neq 0 \text{ and for } (x,y,z,w),\ c_{0,0} = 1/2 \\ \alpha_5 \in \mathbb{Z} \text{ if } \alpha_4 \neq 0 \text{ and for } (x,y,z,w),\ c_{0,0} = \alpha_5/(-\alpha_4+\alpha_5) \neq 1/2. \end{cases}$$

If $c_{0,0} = 1/2$ for (x,y,z,w), it follows from Proposition 2.6 that $\alpha_4+\alpha_5 = 0$, which implies that $\alpha_i \in \mathbb{Z}$ $(0 \leq i \leq 5)$.

If $c_{0,0} \neq 1/2$ for (x,y,z,w) and $\alpha_4 \neq 0$, we obtain the necessary condition. If $c_{0,0} \neq 1/2$ for (x,y,z,w) and $\alpha_4 = 0$, we have $\alpha_1 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}, \alpha_2+\alpha_3 \in \mathbb{Z}, \alpha_2+\alpha_5 \in \mathbb{Z}$. We may then assume that $\alpha_0+\alpha_1 \neq 1$ and $\alpha_2(\alpha_1+\alpha_2) \neq 0$. Therefore, it follows that $s_1s_2(x,y,z,w)$ is a rational solution of $D_5^{(1)}(\alpha_0+\alpha_2, -\alpha_1-\alpha_2, \alpha_1, \alpha_3+\alpha_2, \alpha_4, \alpha_5)$ such that both of $s_1s_2(y,w)$ have a pole at $t=\infty$ and both of $s_1s_2(x,z)$ are holomorphic at

$t=\infty$ and $(a_{\infty,0},c_{\infty,0})=(0,0)$. Moreover, all of $s_1s_2(x,y,z,w)$ are holomorphic at $t=0$ and $b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5$. Thus, considering $a_{\infty,-1}\in\mathbb{Z}$ for $s_1s_2(x,y,z,w)$, we find that $\alpha_i\in\mathbb{Z}\,(0\le i\le 5)$. □

Let us deal with the case where x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5$ and $b_{0,0}\neq 0$.

Proposition 7.4. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type A such that x,y,z,w are all holomorphic at $t=\infty$ and $(a_{\infty,0},c_{\infty,0})=(0,0)$. Moreover, assume that x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5\neq 0$ and $b_{0,0}\neq 0$. Either of the following then occurs:*
(1) $\alpha_0\in\mathbb{Z},\alpha_3\in\mathbb{Z},\alpha_4\in\mathbb{Z},\alpha_1+\alpha_2\in\mathbb{Z}$, (2) $\alpha_0\in\mathbb{Z},\alpha_2\in\mathbb{Z},\alpha_3\in\mathbb{Z},\alpha_5\in\mathbb{Z}$,
(3) $\alpha_1\in\mathbb{Z},\alpha_2\in\mathbb{Z},\alpha_5\in\mathbb{Z},\alpha_3+\alpha_4\in\mathbb{Z}$.

Proof. Considering $a_{\infty,-1}\in\mathbb{Z}$ and $(b_{\infty,0}+d_{\infty,0})-(b_{0,0}+d_{0,0})\in\mathbb{Z}$ for (x,y,z,w), we have $\alpha_2+\alpha_5\in\mathbb{Z}$ and $\alpha_1+\alpha_3+\alpha_4-\alpha_5\in\mathbb{Z}$.

Considering $a_{\infty,-1}\in\mathbb{Z}$ for $s_1(x,y,z,w)$, we can prove that if $\alpha_1\neq 0$, $\alpha_3\in\mathbb{Z}$. Considering $(b_{\infty,0}+d_{\infty,0})-(b_{0,0}+d_{0,0})\in\mathbb{Z}$ for $s_4(x,y,z,w)$, we can show that

$$\begin{cases}\alpha_4\in\mathbb{Z} & \text{if } \alpha_4\neq 0 \text{ and for } (x,y,z,w),\ c_{0,0}=1/2\\ \alpha_5\in\mathbb{Z} & \text{if } \alpha_4\neq 0 \text{ and for } (x,y,z,w),\ c_{0,0}=\alpha_5/(-\alpha_4+\alpha_5)\neq 1/2.\end{cases}$$

If $c_{0,0}=1/2$ and $\alpha_1\neq 0$, we obtain the necessary conditions. If $c_{0,0}=1/2$ and $\alpha_1=0$, we have $\alpha_1\in\mathbb{Z},\alpha_4\in\mathbb{Z},\alpha_2+\alpha_5\in\mathbb{Z},\alpha_2+\alpha_3\in\mathbb{Z}$. We may then assume that $\alpha_0+\alpha_1\neq 1$ and $\alpha_2(\alpha_1+\alpha_2)(\alpha_4+\alpha_5)\neq 0$. Therefore, $s_1s_2(x,y,z,w)$ is a rational solution of $D_5^{(1)}(\alpha_0+\alpha_2,-\alpha_1-\alpha_2,\alpha_1,\alpha_3+\alpha_2,\alpha_4,\alpha_5)$ such that both of $s_1s_2(y,w)$ have a pole at $t=\infty$ and both of $s_1s_2(x,z)$ are holomorphic at $t=\infty$ and $(a_{\infty,0},c_{\infty,0})=(0,0)$. Moreover, all of $s_1s_2(x,y,z,w)$ are holomorphic at $t=0$. Thus, considering $a_{\infty,-1}\in\mathbb{Z}$ for $s_1s_2(x,y,z,w)$, we see that $\alpha_i\in\mathbb{Z}$ $(0\le i\le 5)$.

If $c_{0,0}\neq 1/2$ and $\alpha_1\neq 0$, or $\alpha_4\neq 0$, we can obtain the necessary conditions.

If $c_{0,0}\neq 1/2$ and $\alpha_1=\alpha_4=0$, we have $\alpha_1\in\mathbb{Z},\alpha_4\in\mathbb{Z},\alpha_2+\alpha_5\in\mathbb{Z},\alpha_2+\alpha_3\in\mathbb{Z}$. We may then assume that $\alpha_5\neq 0$. Therefore, $s_5(x,y,z,w)$ is a rational solution of $D_5^{(1)}(\alpha_0,\alpha_1\alpha_2,,\alpha_3+\alpha_5,\alpha_4,-\alpha_5)$ such that both of $s_5(w)$ has a pole at $t=\infty$ and all of $s_5(x,y,z)$ are holomorphic at $t=\infty$ and $(a_{\infty,0},c_{\infty,0})=(0,0)$. Moreover, all of $s_5(x,y,z,w)$ are holomorphic at $t=0$. Considering $(b_{\infty,0}+d_{\infty,0})-(b_{0,0}+d_{0,0})\in\mathbb{Z}$ for $s_5(x,y,z,w)$, we find that $\alpha_i\in\mathbb{Z}$ $(0\le i\le 5)$. □

Let us treat the case where x has a pole at $t=0$.

Proposition 7.5. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type A such that x,y,z,w are all holomorphic at $t=\infty$ and $(a_{\infty,0},c_{\infty,0})=(0,0)$. Moreover,*

assume that x has a pole at $t=0$ and y,z,w are all holomorphic at $t=0$. One of the following then occurs:
(1) $\alpha_1 \in \mathbb{Z}, \alpha_3 \in \mathbb{Z}, \alpha_5 \in \mathbb{Z}, \alpha_0+\alpha_2 \in \mathbb{Z}$, (2) $\alpha_1 \in \mathbb{Z}, \alpha_5 \in \mathbb{Z}, \alpha_0+\alpha_2 \in \mathbb{Z}, \alpha_3+\alpha_4 \in \mathbb{Z}$.

Proof. Considering $a_{\infty,-1}-a_{0,-1} \in \mathbb{Z}$, we first have $\alpha_0-\alpha_1+\alpha_2+\alpha_5 \in \mathbb{Z}$. We next prove that if $\alpha_1 \neq 0$, $\alpha_1+\alpha_2+2\alpha_3+\alpha_4 \in \mathbb{Z}$. If $\alpha_1 \neq 0$, $s_1(x,y,z,w)$ is a solution of $D_5^{(1)}(\alpha_0,-\alpha_1,\alpha_2+\alpha_1,\alpha_3,\alpha_4,\alpha_5)$ such that all of $s_1(x,y,z,w)$ are holomorphic at $t=\infty$ and $(a_{\infty,0},c_{\infty,0})=(1,0)$. Moreover, all of $s_1(x,y,z,w)$ are holomorphic at $t=0$. Considering $a_{\infty,-1}-a_{0,-1} \in \mathbb{Z}$ for $s_1(x,y,z,w)$, we obtain $\alpha_1+\alpha_2+2\alpha_3+\alpha_4 \in \mathbb{Z}$.

Considering $(b_{\infty,0}+d_{\infty,0})-(b_{0,0}+d_{0,0}) \in \mathbb{Z}$, we have

$$\begin{cases} \alpha_1+\alpha_3 \in \mathbb{Z} & \text{if } d_{0,0}=0, \\ \alpha_1+\alpha_3+\alpha_4-\alpha_5 \in \mathbb{Z} & \text{if } d_{0,0}=-\alpha_4+\alpha_5 \neq 0. \end{cases}$$

Moreover, if $\alpha_5 \neq 0$, considering $(b_{\infty,0}+d_{\infty,0})-(b_{0,0}+d_{0,0}) \in \mathbb{Z}$ for $s_5(x,y,z,w)$, we obtain

$$\begin{cases} \alpha_1+\alpha_3-\alpha_5 \in \mathbb{Z} & \text{if for } (x,y,z,w),\ d_{0,0}=0, \\ \alpha_0+2\alpha_2+\alpha_3+\alpha_5 \in \mathbb{Z} & \text{if for } (x,y,z,w),\ d_{0,0}=-\alpha_4+\alpha_5 \neq 0. \end{cases}$$

If $d_{0,0}=0$ for (x,y,z,w), we have $\alpha_1 \in \mathbb{Z}, \alpha_3 \in \mathbb{Z}, \alpha_5 \in \mathbb{Z}, \alpha_0+\alpha_2 \in \mathbb{Z}$. If $d_{0,0}=-\alpha_4+\alpha_5 \neq 0$ for (x,y,z,w), we obtain $\alpha_1 \in \mathbb{Z}, \alpha_5 \in \mathbb{Z}, \alpha_0+\alpha_2 \in \mathbb{Z}, \alpha_3+\alpha_4 \in \mathbb{Z}$. □

Let us deal with the case where z has a pole at $t=0$. By s_1, s_4, we can then show the following proposition:

Proposition 7.6. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a rational solution of type A such that x,y,z,w are all holomorphic at $t=\infty$ and $(a_{\infty,0},c_{\infty,0})=(0,0)$. Moreover, assume that z has a pole at $t=0$ and x,y,w are all holomorphic at $t=0$. Then,*

$$\alpha_1 \in \mathbb{Z}, \alpha_3 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}, \alpha_0+\alpha_2 \in \mathbb{Z}.$$

Proof. Considering $a_{\infty,-1} \in \mathbb{Z}$ and $(b_{\infty,0}+d_{\infty,0})-(b_{0,0}+d_{0,0}) \in \mathbb{Z}$, we have $\alpha_2+\alpha_5 \in \mathbb{Z}$ and $\alpha_1+\alpha_3 \in \mathbb{Z}$.

If $\alpha_1 \neq 0$, considering $a_{\infty,-1}-a_{0,-1} \in \mathbb{Z}$ for $s_1(x,y,z,w)$, we obtain $\alpha_0+\alpha_2+\alpha_4 \in \mathbb{Z}$. If $\alpha_4 \neq 0$, considering $(b_{\infty,0}+d_{\infty,0})-(b_{0,0}+d_{0,0}) \in \mathbb{Z}$ for $s_4(x,y,z,w)$, we have $\alpha_4 \in \mathbb{Z}$.

Thus, we can obtain the necessary condition. □

Let us treat the case where x,z have a pole at $t=0$. By s_1, s_4, we can then show the following proposition:

Proposition 7.7. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a rational solution of type A such that x, y, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. Moreover, assume that x, z both have a pole at $t = 0$ and y, w are both holomorphic at $t = 0$. One of the following then occurs:*
(1) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z}$, (2) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}$.

Proof. We treat the case where z has a pole of order $n\,(n \geq 2)$ at $t = 0$. If z has a pole of order one at $t = 0$, we can prove the proposition in the same way.

Considering $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$ and $(b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}$, we have $\alpha_0 - \alpha_1 + \alpha_2 + \alpha_5 \in \mathbb{Z}$ and $\alpha_1 + \alpha_3 \in \mathbb{Z}$.

If $\alpha_1 \neq 0$, considering $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$ for $s_1(x, y, z, w)$, we have $\alpha_0 \in \mathbb{Z}$. If $\alpha_4 \neq 0$, considering $(b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}$ for $s_4(x, y, z, w)$, we obtain $\alpha_4 \in \mathbb{Z}$.

Therefore, we can obtain the necessary conditions. □

Let us deal with the case where y, w have a pole at $t = 0$.

Proposition 7.8. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a rational solution of type A such that x, y, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. Moreover, assume that y, w both have a pole at $t = 0$ and x, z are both holomorphic at $t = 0$. Then,*

$$\alpha_0 \in \mathbb{Z},\ \alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}.$$

Proof. Considering $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$ and $(b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}$, we have $\alpha_2 + \alpha_5 \in \mathbb{Z}$ and $\alpha_1 + \alpha_3 + \alpha_4 - \alpha_5 \in \mathbb{Z}$.

If $\alpha_0 \neq 0$, considering $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$ for $s_0(x, y, z, w)$, we have $\alpha_0 \in \mathbb{Z}$. If $\alpha_4 \neq 0$, considering $(b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}$ for $s_4(x, y, z, w)$, we have $\alpha_4 \in \mathbb{Z}$.

Therefore, by Proposition 2.21, we can obtain the necessary conditions. □

Let us summarize necessary conditions for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ to have a rational solution of type A such that x, y, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$.

Proposition 7.9. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a rational solution of type A such that x, y, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. One of the following then occurs:*

(1) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$, (2) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$,
(3) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}$, (4) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z}$,
(5) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z}$, (6) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z}$,
(7) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z}$, (8) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z}$,
(9) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z}$, (10) $\alpha_1 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z}$.

7.1.2 The case where $(a_{\infty,0}, c_{\infty,0}) = (1,0)$

Proposition 7.10. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type A such that x, y, z, w are all holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1,0)$. One of the following then occurs:*

(1) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$,
(2) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z}$,
(3) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}$,
(4) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z}$,
(5) $\alpha_0 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$,
(6) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$,
(7) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}$,
(8) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z}$,
(9) $\alpha_0 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$,
(10) $\alpha_1 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$.

Proof. If $\alpha_1 \neq 0$, $s_1(x,y,z,w)$ is a rational solution of type A of $D_5^{(1)}(\alpha_0, -\alpha_1, \alpha_2+\alpha_1, \alpha_3, \alpha_4, \alpha_5)$ such that all of $s_1(x,y,z,w)$ are holomorphic at $t=\infty$ and $(a_{\infty,}, c_{\infty,0}) = (1,0)$. Thus, the proposition follows from Proposition 7.9.

Let us assume that $\alpha_1 = 0$ and $\alpha_2 \neq 0$. $s_2(x,y,z,w)$ is a then rational solution of type A of $D_5^{(1)}(\alpha_0+\alpha_2, \alpha_2, -\alpha_2, \alpha_3+\alpha_2, \alpha_4, \alpha_5)$ such that all of $s_2(x,y,z,w)$ are holomorphic at $t=\infty$ and $(a_{\infty,}, c_{\infty,0}) = (1,0)$. Thus, by the above discussion, we can obtain the necessary conditions.

Let us assume that $\alpha_1 = \alpha_2 = 0$ and $\alpha_0 \neq 0$. By s_0 and the above discussion, we can then obtain the necessary conditions.

Let us assume that $\alpha_0 = \alpha_1 = \alpha_2 = 0$ and $\alpha_3 \neq 0$. Then, by $\pi_3 s_3$ and Proposition 7.9, we can obtain the necessary conditions.

Let us assume that $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4 \neq 0$. By s_4 and the above discussion, we can then obtain the necessary conditions. Furthermore, if $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, the proposition is obvious. ☐

7.1.3 The case where $(a_{\infty,0}, c_{\infty,0}) = (0,1)$

By π_3 and Proposition 7.10, we can prove the following proposition.

Proposition 7.11. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type A such that x, y, z, w are all holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0,1)$. One of the following then occurs:*

(1) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}$,
(2) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z}$,
(3) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$,
(4) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z}$,
(5) $\alpha_0 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$,
(6) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$,
(7) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$,
(8) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z}$,
(9) $\alpha_0 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$,
(10) $\alpha_1 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$.

7.1.4 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$

By π_3 and Proposition 7.9, we can show the following proposition.

Proposition 7.12. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type A such that x, y, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$. One of the following then occurs:*

(1) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}$, (2) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}$,
(3) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$, (4) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z}$,
(5) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z}$, (6) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z}$,
(7) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z}$, (8) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z}$,
(9) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z}$, (10) $\alpha_1 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z}$.

7.2 The case where y has a pole at $t = \infty$

7.2.1 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$

By π_1 and Proposition 7.12, we can prove the following proposition.

Proposition 7.13. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type A such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. One of the following then occurs:*

(1) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$, (2) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$,
(3) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}$, (4) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z}$,
(5) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z}$, (6) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z}$,
(7) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z}$, (8) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z}$,
(9) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z}$, (10) $\alpha_0 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z}$.

7.2.2 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$

By π_1 and Proposition 7.11, we can prove the following proposition.

Proposition 7.14. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type A such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and*

$(a_{\infty,0}, c_{\infty,0}) = (1, 0)$. *One of the following then occurs:*

(1) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$	(2) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},$
(3) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$	(4) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},$
(5) $\alpha_1 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$	(6) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$
(7) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$	(8) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},$
(9) $\alpha_1 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$	(10) $\alpha_0 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z}.$

7.2.3 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$

By π_1 and Proposition 7.10, we can obtain the following proposition:

Proposition 7.15. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a rational solution of type A such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$. One of the following then occurs:*

(1) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$	(2) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},$
(3) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$	(4) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},$
(5) $\alpha_1 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$	(6) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$
(7) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$	(8) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},$
(9) $\alpha_1 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$	(10) $\alpha_0 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z}.$

7.2.4 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$

By π_1 and Proposition 7.9, we can prove the following proposition:

Proposition 7.16. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a rational solution of type A such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$. One of the following then occurs:*

(1) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$	(2) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$
(3) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$	(4) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},$
(5) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$	(6) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$
(7) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},$	(8) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},$
(9) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$	(10) $\alpha_0 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z}.$

7.3 The case where w has a pole at $t = \infty$

7.3.1 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$

By π_2 and Proposition 7.15, we can prove the following proposition:

Proposition 7.17. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$*, there exists a rational solution of type A such that* w *has a pole at* $t = \infty$ *and* x, y, z *are all holomorphic at* $t = \infty$ *and* $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$*. One of the following then occurs:*

(1) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$
(2) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$
(3) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$
(4) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$
(5) $\alpha_1 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$
(6) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},$
(7) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$
(8) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$
(9) $\alpha_0 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$
(10) $\alpha_1 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z}.$

7.3.2 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$

By using s_5 in the same way as Proposition 7.10, we can prove the following proposition.

Proposition 7.18. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$*, there exists a rational solution of type A such that* w *has a pole at* $t = \infty$ *and* x, y, z *are all holomorphic at* $t = \infty$ *and* $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$*. One of the following then occurs:*

(1) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$
(2) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},$
(3) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$
(4) $\alpha_1 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$
(5) $\alpha_0 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$
(6) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$
(7) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$
(8) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},$
(9) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},$
(10) $\alpha_1 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z}.$

7.3.3 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$

By using π_3 and Proposition 7.18, we can prove the following proposition:

Proposition 7.19. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$*, there exists a rational solution of type A such that* w *has a pole at* $t = \infty$ *and* x, y, z *are all holomorphic at* $t = \infty$ *and*

$(a_{\infty,0}, c_{\infty,0}) = (0, 1)$. *One of the following then occurs:*

(1) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$ (2) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},$
(3) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$ (4) $\alpha_1 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$
(5) $\alpha_0 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$ (6) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$
(7) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$ (8) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},$
(9) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},$ (10) $\alpha_1 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z}.$

7.3.4 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$

By using π_3 and Proposition 7.17, we can prove the following proposition:

Proposition 7.20. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a rational solution of type A such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$. One of the following then occurs:*

(1) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$ (2) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$
(3) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$ (4) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$
(5) $\alpha_1 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$ (6) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},$
(7) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$ (8) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$
(9) $\alpha_0 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$ (10) $\alpha_1 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z}.$

7.4 The case where y, w have a pole at $t = \infty$

7.4.1 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$

By π_4 and Proposition 7.17, we can show the following proposition:

Proposition 7.21. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a rational solution of type A such that y, w both have a pole at $t = \infty$ and x, z are both holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. One of the following then occurs:*

(1) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$ (2) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$
(3) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$ (4) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$
(5) $\alpha_0 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$ (6) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},$
(7) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$ (8) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},$
(9) $\alpha_1 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},$ (10) $\alpha_0 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1 + \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z}.$

7.4.2 The case where $(a_{\infty,0}, c_{\infty,0}) = (1,0)$

By π_4 and Proposition 7.18, we can show the following proposition:

Proposition 7.22. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type A such that y, w both have a pole at $t=\infty$ and x, z are both holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1,0)$. One of the following then occurs:*

(1) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$,
(2) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z}$,
(3) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$,
(4) $\alpha_0 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$,
(5) $\alpha_1 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$,
(6) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$,
(7) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$,
(8) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z}$,
(9) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z}$,
(10) $\alpha_0 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$.

7.4.3 The case where $(a_{\infty,0}, c_{\infty,0}) = (0,1)$

By π_4 and Proposition 7.19, we can show the following proposition:

Proposition 7.23. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type A such that y, w both have a pole at $t=\infty$ and x, z are both holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0,1)$. One of the following then occurs:*

(1) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}$,
(2) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z}$,
(3) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$,
(4) $\alpha_0 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$,
(5) $\alpha_1 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$,
(6) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$,
(7) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$,
(8) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z}$,
(9) $\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z}$,
(10) $\alpha_0 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$.

7.4.4 The case where $(a_{\infty,0}, c_{\infty,0}) = (1,1)$

By π_4 and Proposition 7.20, we can show the following proposition:

Proposition 7.24. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type A such that y, w both have a pole at $t=\infty$ and x, z are both holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1,1)$. One of the following then occurs:*

(1) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}$,
(2) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$,
(3) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}$,
(4) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$,
(5) $\alpha_0 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$,
(6) $\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z}$,
(7) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}$,
(8) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$,
(9) $\alpha_1 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},\ \alpha_0+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z}$,
(10) $\alpha_0 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},\ \alpha_1+\alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z}$.

8 Necessary conditions for type B

In this section, we obtain necessary conditions for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ to have a rational solution of type B, that is, a solution such that $b_{\infty,1}+d_{\infty,1}=-1/2$. For this purpose, we use the formulas,

$$a_{\infty,-1}-a_{0,-1}\in\mathbb{Z},\ (b_{\infty,0}+d_{\infty,0})-(b_{0,0}+d_{0,0})\in\mathbb{Z},\ h_{\infty,0}-h_{0,0}\in\mathbb{Z}.$$

8.1 The case where y has a pole at $t=\infty$

8.1.1 The case where x,y,z,w are holomorphic at $t=0$

From Proposition 1.49 and 2.22, it follows that $b_{\infty,0}+d_{\infty,0}=0,-\alpha_4+\alpha_5$ and $b_{0,0}+d_{0,0}=0,-\alpha_4+\alpha_5$. Thus, we consider the following four cases:
(1) $b_{\infty,0}+d_{\infty,0}=-\alpha_4+\alpha_5$ and $b_{0,0}+d_{0,0}=0$,
(2) $b_{\infty,0}+d_{\infty,0}=0$ and $b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5$,
(3) $b_{\infty,0}+d_{\infty,0}=-\alpha_4+\alpha_5\neq 0$ and $b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5$,
(4) $b_{\infty,0}+d_{\infty,0}=0$ and $b_{0,0}+d_{0,0}=0$.
From Proposition 1.10, 1.16 and 3.9, considering $a_{\infty,-1}-a_{0,-1}\in\mathbb{Z}$, we find that $-\alpha_0+\alpha_1\in\mathbb{Z}$.

Proposition 8.1. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. Moreover, assume that x,y,z,w are all holomorphic at $t=0$ and one of the following occurs:*
(1) $b_{\infty,0}+d_{\infty,0}=-\alpha_4+\alpha_5$ *and* $b_{0,0}+d_{0,0}=0$,
(2) $b_{\infty,0}+d_{\infty,0}=0$ *and* $b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5$.
Then,

$$-\alpha_0+\alpha_1\in\mathbb{Z},\ -\alpha_4+\alpha_5\in\mathbb{Z}.$$

Proof. From Proposition 1.10, 1.16 and 3.9, we find that $-\alpha_0+\alpha_1\in\mathbb{Z}$. Moreover, from the assumption and Proposition 3.9 and the residue theorem, we observe that $-\alpha_4+\alpha_5\in\mathbb{Z}$, which proves the proposition. □

Proposition 8.2. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. Moreover, assume that x,y,z,w are all holomorphic at $t=0$ and*

$$b_{\infty,0}+d_{\infty,0}=-\alpha_4+\alpha_5\neq 0,\quad b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5,\ b_{0,0}\neq 0.$$

Then,

$$-\alpha_0+\alpha_1\in\mathbb{Z},\ \alpha_0+\alpha_1\in\mathbb{Z}.$$

Proof. Since $-\alpha_0+\alpha_1 \in \mathbb{Z}$, we can assume that $\alpha_0 \neq 0$. $s_0(x,y,z,w)$ is then a rational solution of type B of $D_5^{(1)}(-\alpha_0,\alpha_1,\alpha_2+\alpha_0,\alpha_3,\alpha_4,\alpha_5)$ such that $s_0(y)$ has a pole at $t=\infty$ and $s_0(x), s_0(z), s_0(w)$ are all holomorphic at $t=\infty$. Furthermore, all of $s_0(x,y,z,w)$ are holomorphic at $t=0$. Thus, since $a_{\infty,-1}-a_{0,-1} \in \mathbb{Z}$, it follows from Proposition 1.10, 1.16 and 3.9 that

$$\alpha_0+\alpha_1 = -(-\alpha_0)+\alpha_1 = -\mathrm{Res}_{t=\infty} s_0(x) \in \mathbb{Z},$$

which the proposition. □

Proposition 8.3. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. Moreover, assume that x,y,z,w are all holomorphic at $t=0$ and*

$$b_{\infty,0}+d_{\infty,0} = -\alpha_4+\alpha_5 \neq 0, \quad b_{0,0}+d_{0,0} = -\alpha_4+\alpha_5, \ b_{0,0}=0, \ c_{0,0}=1/2.$$

One of the following then occurs:
(1) $-\alpha_0+\alpha_1 \in \mathbb{Z}, \ 2\alpha_3+\alpha_4+\alpha_5=0$, (2) $-\alpha_0+\alpha_1 \in \mathbb{Z}, \ \alpha_4+\alpha_5=0$.

Proof. We can assume that $2\alpha_3+\alpha_4+\alpha_5 \neq 0$. Since $b_{0,0}=0$ and $c_{0,0}=1/2$, it follows from Proposition 2.6 that $\alpha_4+\alpha_5=0$. □

Proposition 8.4. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. Moreover, assume that x,y,z,w are all holomorphic at $t=0$ and*

$$b_{\infty,0}+d_{\infty,0} = -\alpha_4+\alpha_5 \neq 0, \quad b_{0,0}+d_{0,0} = -\alpha_4+\alpha_5, \ b_{0,0}=0, \ c_{0,0}\neq 1/2.$$

Either of the following then occurs:
(1) $-\alpha_0+\alpha_1 \in \mathbb{Z}, \ 2\alpha_3+\alpha_4+\alpha_5 \in \mathbb{Z}$, (2) $-\alpha_0+\alpha_1 \in \mathbb{Z}, \ \alpha_4+\alpha_5 \in \mathbb{Z}$.

Proof. We assume that $-\alpha_0+\alpha_1 \in 2\mathbb{Z}$. If $(-\alpha_0+\alpha_1)-1 \in 2\mathbb{Z}$, the proposition can be proved in the same way.

For the proof, we consider the following two cases: (1) $c_{\infty,0}=1/2$, (2) $c_{\infty,0}\neq 1/2$. We first deal with case (1). For this purpose, we use $h_{\infty,0}-h_{0,0} \in \mathbb{Z}$. From Proposition 4.5 and 4.10, it follows that

$$\frac{1}{4}(-\alpha_4+\alpha_5)^2+\alpha_4\alpha_5 \in \mathbb{Z}.$$

$\pi_2(x,y,z,w)$ is a rational solution of $D_5^{(1)}(\alpha_5,\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_0)$ of type B such that $\pi_2(y)$ has a pole at $t=\infty$ and all of $\pi_2(x,z,w)$ are holomorphic at $t=\infty$. Moreover, $\pi_2(x)$

has a pole at $t=0$ and all of $\pi_2(y,z,w)$ are holomorphic at $t=0$ and $c_{\infty,0}=1/2$ and $d_{0,0}=0$. It then follows from Proposition 4.5 and 4.11, that

$$\frac{1}{4}(-\alpha_4+\alpha_5)^2-\frac{1}{2}(\alpha_4+\alpha_5)+\alpha_4\alpha_5 \in \mathbb{Z}.$$

Therefore, we find that $\alpha_4+\alpha_5 \in 2\mathbb{Z}$.

Let us treat case (2). From Proposition 4.6 and 4.10, it follows that

$$\frac{1}{4}(2\alpha_3+\alpha_4+\alpha_5)^2 \in \mathbb{Z}.$$

$\pi_2(x,y,z,w)$ is a rational solution of $D_5^{(1)}(\alpha_5,\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_0)$ of type B such that either of the following occurs:
(i) $\pi_2(w)$ has a pole at $t=\infty$ and all of $\pi_2(x,y,z)$ are holomorphic at $t=\infty$,
(ii) both of $\pi_2(y,w)$ have a pole of order one at $t=\infty$ and both of $\pi_2(x,z)$ are holomorphic at $t=\infty$.
Moreover, $\pi_2(x)$ has a pole at $t=0$ and all of $\pi_2(y,z,w)$ are holomorphic at $t=0$. Now, we suppose that case (ii) occurs. For $\pi_2(x,y,z,w)$, case (1) then occurs in Proposition 1.41 and $d_{0,0}=0$. From Proposition 4.9 and 4.11, it follow that

$$\frac{1}{4}(2\alpha_3+\alpha_4+\alpha_5)^2-\frac{1}{2}(2\alpha_3+\alpha_4+\alpha_5) \in \mathbb{Z}.$$

Therefore, we find that $2\alpha_3+\alpha_4+\alpha_5 \in 2\mathbb{Z}$.

If case (i) occurs, we can obtain the necessary condition in the same way. □

We summarize Propositions 8.2, 8.3 and 8.4.

Proposition 8.5. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and x,y,z,w are all holomorphic at $t=0$. Moreover, assume that*

$$b_{\infty,0}+d_{\infty,0}=-\alpha_4+\alpha_5\neq 0, \quad b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5.$$

The parameters then satisfy one of the following conditions:
(1) $-\alpha_0+\alpha_1 \in \mathbb{Z}$, $-\alpha_4-\alpha_5 \in \mathbb{Z}$, (2) $-\alpha_0+\alpha_1 \in \mathbb{Z}$, $-\alpha_0-\alpha_1 \in \mathbb{Z}$,
(3) $-\alpha_0+\alpha_1 \in \mathbb{Z}$, $-2\alpha_3-\alpha_4-\alpha_5 \in \mathbb{Z}$.

We treat the case where $b_{\infty,0}+d_{\infty,0}=b_{0,0}+d_{0,0}=0$.

Proposition 8.6. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. Moreover, assume that x,y,z,w are all holomorphic at $t=0$ and*

$$b_{\infty,0}+d_{\infty,0}=0, \quad b_{0,0}+d_{0,0}=0.$$

The parameters then satisfy one of the following conditions:
(1) $-\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_0-\alpha_1 \in \mathbb{Z}$, (2) $-\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z}$,
(3) $-\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_0-\alpha_1-2\alpha_2 \in \mathbb{Z}$.

Proof. Since $a_{\infty,-1}-a_{0,-1} \in \mathbb{Z}$, we find that $-\alpha_0+\alpha_1 \in \mathbb{Z}$. If $b_{0,0} \neq 0$, we have $-\alpha_0-\alpha_1 \in \mathbb{Z}$ in the same way as Proposition 8.5. We then assume that $b_{0,0}=d_{0,0}=0$. Thus, it follows from Proposition 1.16 and 2.3 that $c_{\infty,0}=c_{0,0}=\dfrac{\alpha_5}{\alpha_4+\alpha_5}$.

If $\alpha_4 \neq 0$, $s_4(x,y,z,w)$ is a rational solution of $D_5^{(1)}(\alpha_0,\alpha_1,\alpha_2,\alpha_3+\alpha_4,-\alpha_4,\alpha_5)$ such that $s_4(y)$ has a pole at $t=\infty$ and all $s_4(x,z,w)$ are holomorphic at $t=\infty$. Furthermore, all of $s_4(x,y,z,w)$ are holomorphic at $t=0$ and for the constant terms of $s_4(x,y,z,w)$,

$$b_{\infty,0}+d_{\infty,0}=b_{0,0}+d_{0,0}=\alpha_4+\alpha_5=-(-\alpha_4)+\alpha_5.$$

By Proposition 8.5, we obtain the proposition.

If $\alpha_5 \neq 0$, we use s_5 in the same way and can prove the proposition. □

Let us summarize the above discussions and obtain the necessary condition for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ to have a rational solution of type B such that the following holds:
(1) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$
(2) x,y,z,w are all holomorphic at $t=0$.

Proposition 8.7. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. Moreover, assume that x,y,z,w are all holomorphic at $t=0$. One of the following then occurs:*
(1) $-\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z}$, (2) $-\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z}$,
(3) $-\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_0-\alpha_1 \in \mathbb{Z}$, (4) $-\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_0-\alpha_1-2\alpha_2 \in \mathbb{Z}$.

8.1.2 The case where x has a pole at $t=0$

Proposition 8.8. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. Moreover, assume that x has a pole at $t=0$ and y,z,w are all holomorphic at $t=0$. One of the following then occurs:*
(1) $-\alpha_0-\alpha_1 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z}$, (2) $-\alpha_0-\alpha_1 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z}$,
(3) $-\alpha_0-\alpha_1 \in \mathbb{Z},\ -\alpha_0+\alpha_1 \in \mathbb{Z}$, (4) $-\alpha_0-\alpha_1 \in \mathbb{Z},\ -\alpha_0-\alpha_1-2\alpha_2 \in \mathbb{Z}$.

Proof. Since $\mathrm{Res}_{t=0}x=-\alpha_0+\alpha_1$, it follows that $\alpha_1 \neq 0$, or $\alpha_0 \neq 0$.

If $\alpha_1 \neq 0$, $s_1(x,y,z,w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_0,-\alpha_1,\alpha_2+\alpha_1,\alpha_3,\alpha_4,\alpha_5)$ such that $s_1(y)$ has a pole at $t=\infty$ and all of $s_1(x,z,w)$ are holomorphic at $t=\infty$. Moreover, all of $s_1(x,y,z,w)$ are holomorphic at $t=0$. From Proposition 8.7, we then obtain the necessary conditions.

If $\alpha_1=0$ and $\alpha_0 \neq 0$. we use $s_1\pi_4$ in the same way and can prove the proposition. □

8.1.3 The case where z has a pole at $t=0$

Proposition 8.9. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. Moreover, assume that z has a pole at $t=0$ and x,y,w are all holomorphic at $t=0$. One of the following then occurs:*
(1) $-\alpha_0+\alpha_1\in\mathbb{Z},\ -\alpha_4+\alpha_5\in\mathbb{Z}$, (2) $-\alpha_0+\alpha_1\in\mathbb{Z},\ -2\alpha_3-\alpha_4-\alpha_5\in\mathbb{Z}$,
(3) $-\alpha_0+\alpha_1\in\mathbb{Z},\ -\alpha_0-\alpha_1\in\mathbb{Z}$, (4) $-\alpha_0+\alpha_1\in\mathbb{Z},\ -\alpha_4-\alpha_5\in\mathbb{Z}$.

Proof. Since $a_{\infty,-1}-a_{0,-1}\in\mathbb{Z}$, we note that $-\alpha_0+\alpha_1\in\mathbb{Z}$. If $\alpha_3\neq 0$, then $s_3(x,y,z,w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_0,\alpha_1,\alpha_2+\alpha_3,-\alpha_3,\alpha_4+\alpha_3,\alpha_5+\alpha_3)$ such that all of $s_3(x,y,z,w)$ are holomorphic at $t=0$. Moreover, either of the following occurs:
(1) $s_3(y)$ has a pole at $t=\infty$ and all of $s_3(x,z,w)$ are holomorphic at $t=\infty$,
(2) both of $s_3(y,z)$ have a pole at $t=\infty$ and both of $s_3(x,w)$ are holomorphic at $t=\infty$.

If case (1) occurs, we can prove the proposition from Proposition 8.7. If case (2) occurs, the proposition follows from Proposition 1.31 and its corollaries.

If $\alpha_3=0$, we can assume that $\alpha_4\neq 0$ or $\alpha_5\neq 0$. When $\alpha_3=0$ and $\alpha_4\neq 0$, we use s_3s_4 in the same way and can prove the proposition. When $\alpha_3=0$ and $\alpha_5\neq 0$, we use s_3s_5 in the same way and can show the proposition. □

8.1.4 The case in which x,z have a pole at $t=0$

Proposition 8.10. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. Moreover, assume that x has a pole of order one at $t=0$ and z has a pole of order n $(n\ge 2)$ at $t=0$ and y,w are both holomorphic at $t=0$. One of the following then occurs:*
(1) $\alpha_0+\alpha_1\in\mathbb{Z},\ -\alpha_4+\alpha_5\in\mathbb{Z}$, (2) $2\alpha_3+\alpha_4+\alpha_5\in\mathbb{Z},\ -\alpha_4+\alpha_5\in\mathbb{Z}$,
(3) $\alpha_0+\alpha_1\in\mathbb{Z},\ \alpha_4+\alpha_5\in\mathbb{Z}$, (4) $2\alpha_3+\alpha_4+\alpha_5\in\mathbb{Z},\ \alpha_4+\alpha_5\in\mathbb{Z}$.

Proof. From Proposition 2.17, it follows that one of the following occurs:
(i) $\alpha_0+\alpha_1=0$, (ii) $2\alpha_3+\alpha_4+\alpha_5=1$, (iii) $\alpha_3=\alpha_4=\alpha_5=0$.
Let us consider the following two cases: (a) $b_{\infty,0}+d_{\infty,0}=-\alpha_4+\alpha_5$, (b) $b_{\infty,0}+d_{\infty,0}=0$.
If case (a) occurs, we can obtain the necessary conditions, because $b_{0,0}+d_{0,0}=0$.

If case (b) occurs, we can assume that case (2) occurs in Proposition 1.16 and

$$b_{\infty,0}=\frac{(\alpha_4+\alpha_5)(2\alpha_3+\alpha_4+\alpha_5)}{-\alpha_4+\alpha_5},\ c_{\infty,0}=\frac{\alpha_5}{\alpha_4+\alpha_5}\neq\frac{1}{2},\ d_{\infty,0}=-\frac{(\alpha_4+\alpha_5)(2\alpha_3+\alpha_4+\alpha_5)}{-\alpha_4+\alpha_5},$$

which implies that $\alpha_4\neq 0$ or $\alpha_5\neq 0$. If $\alpha_4\neq 0$, $s_4(x,y,z,w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_0,\alpha_1,\alpha_2,\alpha_3+\alpha_4,-\alpha_4,\alpha_5)$ such that $s_4(y)$ has a pole at $t=\infty$ and all of

$s_4(x,z,w)$ are holomorphic at $t=\infty$. Moreover, assume that both of $s_4(x,z)$ have a pole at $t=0$ and both of $s_4(y,w)$ are holomorphic at $t=0$ and

$$b_{\infty,0}+d_{\infty,0}=-(-\alpha_4)+\alpha_5,\ b_{0,0}+d_{0,0}=0.$$

Thus, it follows that $\alpha_0+\alpha_1\in\mathbb{Z},\ \alpha_4+\alpha_5\in\mathbb{Z}$, or $2\alpha_3+\alpha_4+\alpha_5\in\mathbb{Z},\ \alpha_4+\alpha_5\in\mathbb{Z}$.

If $\alpha_5\neq 0$, we use s_5 in the same way and can obtain the necessary conditions. □

Proposition 8.11. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. Moreover, assume that x,z both have a pole of order one at $t=0$ and y,w are both holomorphic at $t=0$ and $b_{0,1}=\alpha_1/(\alpha_0-\alpha_1)$. Either of the following then occurs:*
(1) $\alpha_0+\alpha_1\in\mathbb{Z},\ -\alpha_4+\alpha_5\in\mathbb{Z}$, (2) $\alpha_0+\alpha_1\in\mathbb{Z},\ \alpha_4+\alpha_5\in\mathbb{Z}$.

Proof. Since $\mathrm{Res}_{t=0}x=-\alpha_0+\alpha_1$, it follows that $\alpha_1\neq 0$ or $\alpha_0\neq 0$. If $\alpha_1\neq 0$, then $s_1(x,y,z,w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_0,-\alpha_1,\alpha_2+\alpha_1,\alpha_3,\alpha_4,\alpha_5)$ such that $s_1(y)$ has a pole at $t=\infty$ and all 0f $s_1(x,z,w)$ are holomorphic at $t=\infty$. Moreover, $s_1(z)$ has a pole at $t=0$ and all of $s_1(x,y,w)$ are holomorphic at $t=0$. From the residue theorem, it then follows that

$$-\alpha_0+(-\alpha_1)=-\mathrm{Res}_{t=\infty}s_1(x)\in\mathbb{Z}.$$

If $\alpha_1=0$ and $\alpha_0\neq 0$. we use $s_1\pi_4$ in the same way and can prove that $\alpha_0+\alpha_1\in\mathbb{Z}$. Now, let us consider the following two cases: (i) $b_{\infty,0}+d_{\infty,0}=-\alpha_4+\alpha_5$, (ii) $b_{\infty,0}+d_{\infty,0}=0$.

If case (i) occurs, we find that $-\alpha_4+\alpha_5\in\mathbb{Z}$, because $b_{0,0}+d_{0,0}=0$.

Let us treat the case (ii). For this purpose, we can assume that $\alpha_4\neq 0$ or $\alpha_5\neq 0$ and that case (2) occurs in Proposition 1.16. If $\alpha_4\neq 0$, then $s_4(x,y,z,w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_0,\alpha_1,\alpha_2,\alpha_3+\alpha_4,-\alpha_4,\alpha_5)$ such that $s_4(y)$ has a pole at $t=\infty$ and all of $s_4(x,z,w)$ are holomorphic at $t=\infty$. Moreover, both of $s_4(x,z)$ have a pole at $t=0$ and both of $s_4(y,w)$ are holomorphic at $t=0$ and the constant terms of the Laurent series of $s_4(y,w)$, at $t=\infty,\ 0$ satisfy

$$b_{\infty,0}+d_{\infty,0}=\alpha_4+\alpha_5,\ b_{0,0}+d_{0,0}=0.$$

Thus, it follows that $\alpha_4+\alpha_5\in\mathbb{Z}$.

If $\alpha_5\neq 0$, we use s_5 in the same way and can obtain the necessary conditions. □

Proposition 8.12. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. Moreover, assume that x,z both have a pole of order one at $t=0$ and y,w are both holomorphic at $t=0$ and $c_{\infty,0}=1/2$ and $b_{0,1}=-1/2$. One of the following then occurs:*
(1) $-\alpha_4+\alpha_5\in\mathbb{Z},\ -\alpha_0+\alpha_1\in\mathbb{Z}$, (2) $-\alpha_4+\alpha_5\in\mathbb{Z},\ \alpha_0+\alpha_1\in\mathbb{Z}$,
(3) $-\alpha_4+\alpha_5\in\mathbb{Z},\ \alpha_4+\alpha_5\in\mathbb{Z}$, (4) $-\alpha_4+\alpha_5\in\mathbb{Z},\ 2\alpha_3+\alpha_4+\alpha_5\in\mathbb{Z}$.

Proof. $\pi_2(x, y, z, w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0)$ such that $\pi_2(y)$ has a pole at $t = \infty$ and all of $\pi_2(x, z, w)$ are holomorphic at $t = \infty$. Moreover, all of $\pi_2(x, y, z, w)$ are holomorphic at $t = 0$. Thus, the proposition follows from Proposition 8.7. $\square$

Proposition 8.13. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a rational solution of type B such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$. Moreover, assume that x, z both have a pole of order one at $t = 0$ and y, w are both holomorphic at $t = 0$ and $c_{\infty,0} \neq 1/2$, $b_{0,1} = -1/2$. One of the following then occurs:*

(1) $-\alpha_4 + \alpha_5 \in \mathbb{Z},\ -\alpha_0 + \alpha_1 \in \mathbb{Z}$, (2) $-\alpha_4 + \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_1 \in \mathbb{Z}$,
(3) $-\alpha_4 + \alpha_5 \in \mathbb{Z},\ 2\alpha_3 + \alpha_4 + \alpha_5 \in \mathbb{Z}$, (4) $-\alpha_4 + \alpha_5 \in \mathbb{Z},\ \alpha_4 + \alpha_5 \in \mathbb{Z}$,
(5) $\alpha_4 + \alpha_5 \in \mathbb{Z},\ -\alpha_0 + \alpha_1 \in \mathbb{Z}$, (6) $\alpha_4 + \alpha_5 \in \mathbb{Z},\ \alpha_0 + \alpha_1 \in \mathbb{Z}$,
(7) $\alpha_4 + \alpha_5 \in \mathbb{Z},\ 2\alpha_3 + \alpha_4 + \alpha_5 \in \mathbb{Z}$.

Proof. For the proof, we can assume that case (1) or (2) occurs in Proposition 1.16, which implies that $\alpha_4 \neq 0$ or $\alpha_5 \neq 0$. If case (3) occurs in Proposition 1.16, then $\alpha_4 = \alpha_5 = 0$, which implies that the parameters satisfy one of the conditions in the proposition.

If case (1) occurs in Proposition 1.16, it follows that $b_{\infty,0} + d_{\infty,0} = -\alpha_4 + \alpha_5$ and $b_{0,0} + d_{0,0} = 0$, which implies that $-\alpha_4 + \alpha_5 \in \mathbb{Z}$. We may then assume that $\alpha_4 \neq 0$ and $\alpha_5 \neq 0$.

When $\alpha_3 \neq 0$, $s_3(x, y, z, w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5 + \alpha_3)$ such that $s_3(y)$ has a pole at $t = \infty$ and all of $s_3(x, z, w)$ are holomorphic at $t = \infty$. Moreover, both of $s_3(x, z)$ have a pole at $t = 0$ and both of $s_3(y, w)$ are holomorphic at $t = 0$ and for $s_3(x, y, z, w)$, $c_{\infty,0} = 1/2$. From Proposition 8.12, we can then obtain the necessary conditions. When $\alpha_3 = 0$, we use $s_5 s_4$ in the same way and can obtain the necessary conditions.

If case (2) occurs in Proposition 1.16, we can assume that $\alpha_4 \neq 0$ or $\alpha_5 \neq 0$. When $\alpha_4 \neq 0$, $s_4(x, y, z, w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4, \alpha_5)$ such that $s_4(y)$ has a pole at $t = \infty$ and all of $s_4(x, z, w)$ are holomorphic at $t = \infty$. Moreover, both of $s_4(x, z)$ have a pole at $t = 0$ and both of $s_4(y, w)$ have a pole at $t = 0$ and for $s_4(x, y, z, w)$, case (1) occurs in Proposition 1.16. Then, from the above discussion, we obtain the necessary conditions. When $\alpha_5 \neq 0$, we use s_5 in the same way and can obtain the necessary conditions. $\square$

8.1.5 The case where y, w have a pole at $t = 0$

Proposition 8.14. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a rational solution of type B such that y has a pole at $t = \infty$ and x, z, w are holomorphic at $t = \infty$. Moreover, y, w both have a pole of order n $(n \ge 1)$ at $t = 0$ and x, z are both holomorphic at $t = 0$. One of the following then occurs:*

$$
\begin{array}{llll}
(1) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ \alpha_0+\alpha_1+2\alpha_2=1, & (2) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ \alpha_4+\alpha_5=0,\\
(3) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_4+\alpha_5=0, & (4) & \alpha_0=\alpha_1=\alpha_2=0.
\end{array}
$$

Proof. Since $a_{\infty,-1}-a_{0,-1} \in \mathbb{Z}$, we first note $-\alpha_0+\alpha_1 \in \mathbb{Z}$. Therefore, the proposition follows from Proposition 2.21. □

8.1.6 Summary

Proposition 8.15. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. One of the following then occurs:*

$$
\begin{array}{llll}
(1) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z}, & (2) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z},\\
(3) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_0-\alpha_1 \in \mathbb{Z}, & (4) & -\alpha_0-\alpha_1 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z},\\
(5) & -\alpha_0-\alpha_1 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z}, & (6) & -\alpha_4-\alpha_5 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z},\\
(7) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -2\alpha_3-\alpha_4-\alpha_5 \in \mathbb{Z}, & (8) & -\alpha_0-\alpha_1 \in \mathbb{Z},\ -2\alpha_3-\alpha_4-\alpha_5 \in \mathbb{Z},\\
(9) & -\alpha_0-\alpha_1-2\alpha_2 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z}, & (10) & -\alpha_0-\alpha_1-2\alpha_2 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z}.
\end{array}
$$

8.2 The case where w has a pole at $t=\infty$

Proposition 8.16. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that w has a pole at $t=\infty$ and x,y,z are all holomorphic at $t=\infty$. One of the following then occurs:*

$$
\begin{array}{llll}
(1) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z}, & (2) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z},\\
(3) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_0-\alpha_1 \in \mathbb{Z}, & (4) & -\alpha_0-\alpha_1 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z},\\
(5) & -\alpha_0-\alpha_1 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z}, & (6) & -\alpha_4-\alpha_5 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z},\\
(7) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -2\alpha_3-\alpha_4-\alpha_5 \in \mathbb{Z}, & (8) & -\alpha_0-\alpha_1 \in \mathbb{Z},\ -2\alpha_3-\alpha_4-\alpha_5 \in \mathbb{Z},\\
(9) & -\alpha_0-\alpha_1-2\alpha_2 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z}, & (10) & -\alpha_0-\alpha_1-2\alpha_2 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z}.
\end{array}
$$

Proof. From Proposition 1.26, it follows that $\pi_2(x,y,z,w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_5,\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_0)$ such that $\pi_2(y)$ has a pole at $t=\infty$ and all of $\pi_2(x,z,w)$ are holomorphic at $t=\infty$. Therefore, by considering Proposition 8.15, we can prove the proposition. □

8.3 The case in which y,z have a pole at $t=\infty$

Proposition 8.17. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B such that y,z both have a pole at $t=\infty$ and x,w are both holomorphic at $t=\infty$. Then,*

$-\alpha_4+\alpha_5=0$ *or* $-\alpha_4-\alpha_5=0$. *Furthermore, one of the following occurs:*

$$
\begin{array}{llll}
(1) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z}, & (2) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z},\\
(3) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_0-\alpha_1 \in \mathbb{Z}, & (4) & -\alpha_0-\alpha_1 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z},\\
(5) & -\alpha_0-\alpha_1 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z}, & (6) & -\alpha_4-\alpha_5 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z},\\
(7) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -2\alpha_3-\alpha_4-\alpha_5 \in \mathbb{Z}, & (8) & -\alpha_0-\alpha_1 \in \mathbb{Z},\ -2\alpha_3-\alpha_4-\alpha_5 \in \mathbb{Z},\\
(9) & -\alpha_0-\alpha_1-2\alpha_2 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z}, & (10) & -\alpha_0-\alpha_1-2\alpha_2 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z}.
\end{array}
$$

Proof. If $\alpha_3 \neq 0$, it then follows that $s_3(x,y,z,w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_0,\alpha_1,\alpha_2+\alpha_3,-\alpha_3,\alpha_4+\alpha_3,\alpha_5+\alpha_3)$ such that $s_3(y)$ has a pole at $t=\infty$ and all of $s_3(x,z,w)$ are holomorphic at $t=\infty$. By considering Proposition 8.15, we can obtain the necessary conditions.

If $\alpha_3=0$, we can assume that $\alpha_4 \neq 0$ or $\alpha_5 \neq 0$. When $\alpha_3=0$ and $\alpha_4 \neq 0$, by s_4 and the above discussion, we can obtain the necessary conditions. When $\alpha_3=0$ and $\alpha_5 \neq 0$, by s_5 and the above discussion, we can find the necessary conditions. □

8.4 The case where y,w have a pole at $t=\infty$

8.4.1 The case where y,w have a pole of order one at $t=\infty$

Proposition 8.18. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, *there exists a rational solution of type B such that* y,w *both have a pole of order one at* $t=\infty$ *and* x,z *are both holomorphic at* $t=\infty$. *One of the following then occurs.*

$$
\begin{array}{llll}
(1) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z}, & (2) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z},\\
(3) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -\alpha_0-\alpha_1 \in \mathbb{Z}, & (4) & -\alpha_0-\alpha_1 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z},\\
(5) & -\alpha_0-\alpha_1 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z}, & (6) & -\alpha_4-\alpha_5 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z},\\
(7) & -\alpha_0+\alpha_1 \in \mathbb{Z},\ -2\alpha_3-\alpha_4-\alpha_5 \in \mathbb{Z}, & (8) & -\alpha_0-\alpha_1 \in \mathbb{Z},\ -2\alpha_3-\alpha_4-\alpha_5 \in \mathbb{Z},\\
(9) & -\alpha_0-\alpha_1-2\alpha_2 \in \mathbb{Z},\ -\alpha_4+\alpha_5 \in \mathbb{Z}, & (10) & -\alpha_0-\alpha_1-2\alpha_2 \in \mathbb{Z},\ -\alpha_4-\alpha_5 \in \mathbb{Z}.
\end{array}
$$

Proof. $\pi_2(x,y,z,w)$ is a rational solution of $D_5^{(1)}(\alpha_5,\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_0)$ such that $\pi_2(y)$ has a pole at $t=\infty$ and all of $pi_2(x,z,w)$ are holomorphic at $t=\infty$. Therefore, the proposition follows from Proposition 8.15. □

8.4.2 The case where y,w have a pole of order n $(n \geq 2)$ at $t=\infty$

Proposition 8.19. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, *there exists a rational solution of type B such that* y,w *both have a pole of order* n $(n \geq 2)$ *at* $t=\infty$ *and* x,z *are both*

holomorphic at $t = \infty$. One of the following then occurs:

(1) $-\alpha_0 + \alpha_1 \in \mathbb{Z},\ -\alpha_4 + \alpha_5 \in \mathbb{Z},$ (2) $-\alpha_0 + \alpha_1 \in \mathbb{Z},\ -\alpha_4 - \alpha_5 \in \mathbb{Z},$
(3) $-\alpha_0 + \alpha_1 \in \mathbb{Z},\ -\alpha_0 - \alpha_1 \in \mathbb{Z},$ (4) $-\alpha_0 - \alpha_1 \in \mathbb{Z},\ -\alpha_4 + \alpha_5 \in \mathbb{Z},$
(5) $-\alpha_0 - \alpha_1 \in \mathbb{Z},\ -\alpha_4 - \alpha_5 \in \mathbb{Z},$ (6) $-\alpha_4 - \alpha_5 \in \mathbb{Z},\ -\alpha_4 + \alpha_5 \in \mathbb{Z},$
(7) $-\alpha_0 + \alpha_1 \in \mathbb{Z},\ -2\alpha_3 - \alpha_4 - \alpha_5 \in \mathbb{Z},$ (8) $-\alpha_0 - \alpha_1 \in \mathbb{Z},\ -2\alpha_3 - \alpha_4 - \alpha_5 \in \mathbb{Z},$
(9) $-\alpha_0 - \alpha_1 - 2\alpha_2 \in \mathbb{Z},\ -\alpha_4 + \alpha_5 \in \mathbb{Z},$ (10) $-\alpha_0 - \alpha_1 - 2\alpha_2 \in \mathbb{Z},\ -\alpha_4 - \alpha_5 \in \mathbb{Z}.$

Proof. If $\alpha_2 \neq 0$, it follows that $s_2(x, y, z, w)$ is a rational solution of $D_5^{(1)}(\alpha_0 + \alpha_2, \alpha_1 + \alpha_2 - \alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5)$ such that one of the following occurs:
(1) $s_2(y)$ has a pole at $t = \infty$ and all of $s_2(x, z, w)$ are holomorphic at $t = \infty$,
(2) $s_2(w)$ has a pole at $t = \infty$ and all of $s_2(x, y, z)$ are holomorphic at $t = \infty$,
(3) both of $s_2(y, w)$ have a pole at $t = \infty$ and both of $s_2(x, z)$ are holomorphic at $t = \infty$.
The proposition then follows from Propositions 8.15, 8.16 and 8.18.

If $\alpha_2 = 0$, we may assume that $\alpha_0 \neq 0$ or $\alpha_1 \neq 0$. When $\alpha_2 = 0$ and $\alpha_0 \neq 0$, by s_0, we can obtain the necessary conditions. When $\alpha_2 = 0$ and $\alpha_1 \neq 0$, by s_1, we can obtain the necessary conditions. □

9 Summary and reduction of the necessary conditions

9.1 Summary of the necessary conditions for type A

Theorem 9.1. *Suppose that $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$ has a rational solution of type A. One of the following then occurs:*

(1) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$ (2) $\alpha_0 \in \mathbb{Z},\ \alpha_2 + \alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$
(3) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$ (4) $\alpha_0 \in \mathbb{Z},\ \alpha_2 + \alpha_1 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$
(5) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$ (6) $\alpha_1 \in \mathbb{Z},\ \alpha_2 + \alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$
(7) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_4 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$ (8) $\alpha_1 \in \mathbb{Z},\ \alpha_2 + \alpha_0 \in \mathbb{Z},\ \alpha_4 + \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$
(9) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$ (10) $\alpha_0 \in \mathbb{Z},\ \alpha_2 + \alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$
(11) $\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$ (12) $\alpha_0 \in \mathbb{Z},\ \alpha_2 + \alpha_1 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$
(13) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$ (14) $\alpha_1 \in \mathbb{Z},\ \alpha_2 + \alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$
(15) $\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$ (16) $\alpha_1 \in \mathbb{Z},\ \alpha_2 + \alpha_0 \in \mathbb{Z},\ \alpha_3 + \alpha_5 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}.$

Therefore, by some Bäcklund transformations, the parameters are transformed so that

$$\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z}.$$

9.2 Summary of the necessary conditions for type B

Theorem 9.2. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, there exists a rational solution of type B. The parameters then satisfy one of the following conditions:*

$$
\begin{array}{llll}
(1) & -\alpha_0+\alpha_1\in\mathbb{Z},\ -\alpha_4+\alpha_5\in\mathbb{Z}, & (2) & -\alpha_0+\alpha_1\in\mathbb{Z},\ -\alpha_4-\alpha_5\in\mathbb{Z},\\
(3) & -\alpha_0+\alpha_1\in\mathbb{Z},\ -\alpha_0-\alpha_1\in\mathbb{Z}, & (4) & -\alpha_0-\alpha_1\in\mathbb{Z},\ -\alpha_4+\alpha_5\in\mathbb{Z},\\
(5) & -\alpha_0-\alpha_1\in\mathbb{Z},\ -\alpha_4-\alpha_5\in\mathbb{Z}, & (6) & -\alpha_4-\alpha_5\in\mathbb{Z},\ -\alpha_4+\alpha_5\in\mathbb{Z},\\
(7) & -\alpha_0+\alpha_1\in\mathbb{Z},\ -2\alpha_3-\alpha_4-\alpha_5\in\mathbb{Z}, & (8) & -\alpha_0-\alpha_1\in\mathbb{Z},\ -2\alpha_3-\alpha_4-\alpha_5\in\mathbb{Z},\\
(9) & -\alpha_0-\alpha_1-2\alpha_2\in\mathbb{Z},\ -\alpha_4+\alpha_5\in\mathbb{Z}, & (10) & -\alpha_0-\alpha_1-2\alpha_2\in\mathbb{Z},\ -\alpha_4-\alpha_5\in\mathbb{Z}.
\end{array}
$$

Therefore, by some Bäcklund transformations, the parameters are transformed so that

$$-\alpha_0+\alpha_1\in\mathbb{Z},\ -\alpha_4+\alpha_5\in\mathbb{Z}.$$

9.3 Shift operators

Following Sasano [32], we introduce the shift operators.

Proposition 9.3. *Let T_i $(1\le i\le 6)$ be defined by*

$$
\begin{array}{lll}
T_1 := \pi_1 s_5 s_3 s_2 s_1 s_0 s_2 s_3 s_5, & T_2 := \pi_2 T_1 \pi_2, & T_3 := s_1 s_4 T_1 s_4 s_1,\\
T_4 := s_2 s_3 T_3 s_3 s_2, & T_5 := s_1 T_4 s_1, & T_6 := s_3 T_3 s_3.
\end{array}
$$

Then,

$$
\begin{aligned}
T_1(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)+(0,0,0,0,1,-1),\\
T_2(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)+(-1,1,0,0,0,0),\\
T_3(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)+(0,0,0,1,-1,-1),\\
T_4(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)+(1,1,-1,0,0,0),\\
T_5(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)+(0,0,1,-1,0,0).
\end{aligned}
$$

9.4 Reduction of the parameters for type A

Theorem 9.4. *Suppose that $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ has a rational solution of type A. By some Bäcklund transformations, the parameters can then be transformed so that*

$$\alpha_1=\alpha_2=\alpha_3=\alpha_5=0.$$

9.5 Reduction of the parameters for type B

Theorem 9.5. *Suppose that $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has a rational solution of type B. By some Bäcklund transformations, the parameters can then be transformed so that one of the following occurs:*

$$\mathrm{I}: \quad -\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=0, \quad \mathrm{II}: \quad -\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1.$$

We call cases I *and* II *the standard forms* I *and* II. *Moreover, the parameters are transformed to the standard form* I *if one of the following occurs:*

(1)	$-\alpha_0+\alpha_1\in\mathbb{Z}$,	$-\alpha_4+\alpha_5\in\mathbb{Z}$,	$-\alpha_0+\alpha_1\equiv-\alpha_4+\alpha_5$	mod 2,
(2)	$-\alpha_0+\alpha_1\in\mathbb{Z}$,	$-\alpha_4-\alpha_5\in\mathbb{Z}$,	$-\alpha_0+\alpha_1\equiv-\alpha_4-\alpha_5$	mod 2,
(3)	$-\alpha_0+\alpha_1\in\mathbb{Z}$,	$-\alpha_0-\alpha_1\in\mathbb{Z}$,	$-\alpha_0+\alpha_1\not\equiv-\alpha_0-\alpha_1$	mod 2,
(4)	$-\alpha_0-\alpha_1\in\mathbb{Z}$,	$-\alpha_4+\alpha_5\in\mathbb{Z}$,	$-\alpha_0-\alpha_1\equiv-\alpha_4+\alpha_5$	mod 2,
(5)	$-\alpha_0-\alpha_1\in\mathbb{Z}$,	$-\alpha_4-\alpha_5\in\mathbb{Z}$,	$-\alpha_0-\alpha_1\equiv-\alpha_4-\alpha_5$	mod 2,
(6)	$-\alpha_4-\alpha_5\in\mathbb{Z}$,	$-\alpha_4+\alpha_5\in\mathbb{Z}$,	$-\alpha_4-\alpha_5\not\equiv-\alpha_4+\alpha_5$	mod 2,
(7)	$-\alpha_0+\alpha_1\in\mathbb{Z}$,	$-2\alpha_3-\alpha_4-\alpha_5\in\mathbb{Z}$,	$-\alpha_0+\alpha_1\equiv-2\alpha_3-\alpha_4-\alpha_5$	mod 2,
(8)	$-\alpha_0-\alpha_1\in\mathbb{Z}$,	$-2\alpha_3-\alpha_4-\alpha_5\in\mathbb{Z}$,	$-\alpha_0+\alpha_1\equiv-2\alpha_3-\alpha_4-\alpha_5$	mod 2,
(9)	$-\alpha_0-\alpha_1-2\alpha_2\in\mathbb{Z}$,	$-\alpha_4+\alpha_5\in\mathbb{Z}$,	$-\alpha_0-\alpha_1-2\alpha_2\equiv-\alpha_4+\alpha_5$	mod 2,
(10)	$-\alpha_0-\alpha_1-2\alpha_2\in\mathbb{Z}$,	$-\alpha_4-\alpha_5\in\mathbb{Z}$,	$-\alpha_0-\alpha_1-2\alpha_2\equiv-\alpha_4-\alpha_5$	mod 2.

Otherwise, the parameters are transformed to the standard form II.

10 Rational solutions of type A of $D_5^{(1)}(\alpha_0,0,0,0,\alpha_4,0)$

In this section, we treat the case where $\alpha_1=\alpha_2=\alpha_3=\alpha_5=0$. For this purpose, we have the following lemmas:

Lemma 10.1. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists a rational solution. Moreover, assume that $h_{\infty,0}-h_{0,0}=0$. $a_{\infty,-1}-a_{0,-1}(=-\mathrm{Res}_{t=\infty}x-\mathrm{Res}_{t=0}x)$ is then a non-positive integer.*

Proof. From the residue theorem, we may assume that x has a pole at $t=c\in\mathbb{C}^*$. Since $h_{\infty,0}-h_{0,0}=0$, it follows from Proposition 4.22 that H is holomorphic in $\mathbb{C}^*$. Therefore, it follows from the discussion in Section 4 that one of the following can only occur:
(1) x has a pole at $t=c$ and y,z,w are all holomorphic $t=c$ and $b_{c,0}=0$,
(2) x,z both have a pole at $t=c$ and y,z are both holomorphic at $t=c$ and $(b_{c,0},d_{c,0})=(0,0)$,

(3) x, z both have a pole at $t = c$ and y, z are both holomorphic at $t = c$ and $(b_{c,0}, d_{c,0}) = (-c, c)$, $(a_{c,-1}, c_{c,-1}) = (-1, -1)$,
(4) x, w both have a pole at $t = c$ and y, z are both holomorphic at $t = c$ and $(b_{c,0}, d_{c,0}) = (0, 0), (0, 1)$.
From the discussions in Section 3, we see that when case (1), (2), (3) or (4) occurs, $\mathrm{Res}_{t=c} x = -1$, which proves the lemma. $\square$

Lemma 10.2. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, there exists a rational solution. Moreover, assume that $h_{\infty,0} - h_{0,0} = 0$ and $a_{\infty,-1} - a_{0,-1} (= -\mathrm{Res}_{t=\infty} x - \mathrm{Res}_{t=0} x) = 0$. Then, x is given by*

$$x = a_{\infty,0} + a_{\infty,-1} t^{-1}.$$

Proof. Considering the partial fractional expansion of x, we have only to prove that x is holomorphic in $\mathbb{C}^*$. For this purpose, suppose that x has a pole at $t = c \in \mathbb{C}^*$. From the proof of Lemma 10.1, it follows that $a_{\infty,-1} - a_{0,-1}$ is a negative integer, which is impossible. $\square$

10.1 The case where x, y, z, w are all holomorphic at $t = \infty$

10.1.1 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$

Proposition 10.3. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that x, y, z, w are all holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. Then, $x = y = z = w \equiv 0$.*

Proof. The proposition follows from the direct calculation and Proposition 1.2. $\square$

10.1.2 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$

Proposition 10.4. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that x, y, z, w are all holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$. Then, $\alpha_0, \alpha_4 \in \mathbb{Z}$.*

Proof. We first note that $y \equiv w \equiv 0$ and $a_{\infty,-1} = \alpha_4$. From the discussion in Section 4, we have $h_{\infty,0} - h_{0,0} = 0$.

If x is holomorphic at $t = 0$, it follows from the residue theorem that $\alpha_0 \in \mathbb{Z}$, $\alpha_4 \in \mathbb{Z}$. If x has a pole at $t = 0$, we obtain

$$a_{\infty,-1} - a_{0,-1} = \alpha_4 - (-\alpha_0 + \alpha_1) = \alpha_0 + \alpha_4 = 1,$$

which contradicts Lemma 10.1. $\square$

10.1.3 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$

Proposition 10.5. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that x, y, z, w are all holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. By Proposition 1.1, we first note that $y \equiv w \equiv 0$ and $a_{\infty,-1} = 0$, $c_{\infty,-1} = \alpha_4$. We may next assume that $\alpha_0 \neq 0$ and x is holomorphic at $t = 0$. If x has a pole at $t = 0$, it follows from the residue theorem and Proposition 1.1 that

$$a_{\infty,-1} - a_{0,-1} = 0 - (-\alpha_0 + \alpha_1) = \alpha_0 \in \mathbb{Z}.$$

From the discussions in Section 4, we have $h_{\infty,0} - h_{0,0} = 0$, which implies that $x \equiv 0$ from Lemma 10.2. Since $x \equiv y \equiv w \equiv 0$, it follows from Proposition 2.10 that either of the following occurs:
(1) z is holomorphic at $t = 0$,
(2) z has a pole of order one at $t = 0$ and $c_{0,-1} = \mathrm{Res}_{t=0} z = -\alpha_0$.
From the discussions in Section 3, it follows that either of the following occurs:
(1) z is holomorphic at $t = c \in \mathbb{C}^*$,
(2) z has a pole of order one at $t = c \in \mathbb{C}^*$ and $\mathrm{Res}_{t=c} z = -1$.
If z is holomorphic at $t = 0$, by the residue theorem, we have $\alpha_4 \in \mathbb{Z}$. If z has a pole at $t = 0$, by the residue theorem, we find that

$$c_{\infty,-1} - c_{0,-1} = \alpha_4 - (-\alpha_0) = 1,$$

which is impossible. □

10.1.4 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$

Proposition 10.6. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that x, y, z, w are all holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. By Proposition 1.1, we first note that $y \equiv w \equiv 0$ and $a_{\infty,-1} = \alpha_4$. From the discussions in Section 4, we have $h_{\infty,0} - h_{0,0} = 0$, which implies that $a_{\infty,-1} - a_{0,-1}$ is a non-positive integer from Lemma 10.1.

If x is holomorphic at $t = 0$, we have $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$. If x has a pole at $t = 0$, by the residue theorem, we obtain

$$a_{\infty,-1} - a_{0,-1} = \alpha_4 - (-\alpha_0 + \alpha_1) = 1,$$

which contradicts Lemma 10.1. □

10.2 The case where y has a pole at $t = \infty$

10.2.1 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$

Proposition 10.7. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. By Proposition 1.7, we first note that $w \equiv 0$ and $a_{\infty,-1} = 0$, $b_{\infty,0} = \alpha_0$, $d_{\infty,0} = \alpha_3$. If x has a pole at $t = 0$, considering $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$, we have $\alpha_0 \in \mathbb{Z}$. If $b_{0,0} + d_{0,0} = 0$, considering $(b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}$, we obtain $\alpha_0 \in \mathbb{Z}$.

From the discussions in Section 2, we may assume that x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5 \neq 0$, which implies that $b_{0,0} = -\alpha_4 + \alpha_5$ because $w \equiv 0$. From the discussions in Section 4, it then follows that $h_{\infty,0} - h_{0,0} = 0$, which implies that $x \equiv 0$ from Lemma 10.2. Therefore, y is holomorphic at $t = c \in \mathbb{C}^*$ or has a pole of order one at $t = c$ with $\mathrm{Res}_{t=c} y = -c$. Thus, it follows that y is given by

$$y = -t + b_{\infty,0} + \sum_{k=1}^{n} \frac{-c_k}{t - c_k}, \quad c_k \in \mathbb{C}^*, \ n \in \mathbb{N},$$

which implies that $b_{\infty,0} - b_{0,0}$ is a non-positive integer. On the other hand, by direct calculation, we have

$$b_{\infty,0} - b_{0,0} = \alpha_0 - (-\alpha_4 + \alpha_5) = \alpha_0 + \alpha_4 = 1,$$

which is impossible. □

10.2.2 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$

Proposition 10.8. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. It can be proved in the same way as Proposition 10.7. □

10.2.3 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$

Proposition 10.9. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. The proposition follows from Proposition 7.14. □

10.2.4 The case where $(a_{\infty,0}, c_{\infty,0}) = (1,1)$

Proposition 10.10. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (1,1)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. The proposition follows from Proposition 7.16. □

10.3 The case where w has a pole at $t = \infty$

10.3.1 The case where $(a_{\infty,0}, c_{\infty,0}) = (0,0)$

Proposition 10.11. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (0,0)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. By Proposition 1.23, we first note that $a_{\infty,-1} = c_{\infty,-1} = 0, d_{\infty,0} = \alpha_0$ and $y \equiv 0$. Moreover, we may assume that $\alpha_0 \neq 0, \alpha_4 \neq 0$.

If x has a pole at $t = 0$, considering $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$, we have $\alpha_0 \in \mathbb{Z}$. If $b_{0,0} + d_{0,0} = 0$, considering $(b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}$, we obtain $\alpha_0 \in \mathbb{Z}$. Therefore, from the discussions in Section 2, we may assume that x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5 \neq 0$, which implies that $d_{0,0} = -\alpha_4 + \alpha_5$.

From the discussions in Section 4, we obtain $h_{\infty,0} - h_{0,0} = 0$, which implies that $x \equiv 0$ from Lemma 10.2. From Proposition 3.4, it follows that if z has a pole at $t = c \in \mathbb{C}^*$, z has a pole of order one at $t = c$ and $\mathrm{Res}_{t=c} z = -1$, which implies that $z \equiv 0$ because $c_{\infty,-1} = 0$.

Since $x \equiv y \equiv z \equiv 0$, it follows from the discussion in Section 3 that w has a pole of order at most one at $t = c \in \mathbb{C}^*$ and $\mathrm{Res}_{t=c} w = -c$. We then find that

$$w = -t + d_{\infty,0} + \sum_{k=1}^{n} \frac{-c_k}{t - c_k} \;\; c_k \in \mathbb{C}^*, \;\; n \in \mathbb{N},$$

which shows that $d_{\infty,0} - d_{0,0}$ is a non-positive integer. On the other hand, by direct calculation, we have

$$d_{\infty,0} - d_{0,0} = \alpha_0 - (-\alpha_4 + \alpha_5) = \alpha_0 + \alpha_4 = 1,$$

which is impossible. □

10.3.2 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$

Proposition 10.12. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. The proposition follows from Proposition 7.19. □

10.3.3 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$

Proposition 10.13. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. By Proposition 1.23, we first note that $a_{\infty,-1} = \alpha_4,\ c_{\infty,-1} = 0, d_{\infty,0} = \alpha_0$ and $y \equiv 0$. Moreover, we may assume that $\alpha_0 \neq 0, \alpha_4 \neq 0$.

If x is holomorphic at $t = 0$, considering $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$, we have $\alpha_4 \in \mathbb{Z}$. If $b_{0,0} + d_{0,0} = 0$, considering $(b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}$, we obtain $\alpha_0 \in \mathbb{Z}$. Therefore, from the discussions in Section 2, we may assume that x has a pole at $t = 0$ and $d_{0,0} = -\alpha_4 + \alpha_5 \neq 0$.

From the discussions in Section 4, we have $h_{\infty,0} - h_{0,0} = 0$. On the other hand, by direct calculation, we have

$$a_{\infty,-1} - a_{0,-1} = \alpha_4 - (-\alpha_0 + \alpha_1) = \alpha_0 + \alpha_4 = 1,$$

which contradicts Lemma 10.1. □

10.3.4 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$

Proposition 10.14. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. The proposition follows from Proposition 7.20. □

10.4 The case where y, w have a pole at $t = \infty$

10.4.1 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$

Proposition 10.15. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that y, w both have a pole at $t = \infty$ and x, z are both holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. By Proposition 1.38, we first note that $a_{\infty,-1} = c_{\infty,-1} = 0$ and $b_{\infty,0} = \alpha_0, d_{\infty,0} = -\alpha_0$.

If x has a pole at $t = 0$, considering $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$, we have $\alpha_0 \in \mathbb{Z}$. If $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5$, considering $(b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}$, we obtain $\alpha_4 \in \mathbb{Z}$. Therefore, from the discussions in Section 2, we have only to consider the following two cases:
(1) x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = 0$,
(2) z has a pole of order one at $t = 0$ and x, y, w are all holomorphic at $t = 0$.
Thus, from the discussions in Section 4, it follows that $h_{\infty,0} - h_{0,0} = 0$, which implies that $x \equiv 0$ from Lemma 10.2.

Since $x \equiv 0$, it follows from the discussions in Section 3 that y is holomorphic at $t = c \in \mathbb{C}^*$ or y has a pole order one at $t = c$ with $\mathrm{Res}_{t=c} y = -c$. We then find that

$$y = -t + b_{\infty,0} + \sum_{k=1}^{n} \frac{-c_k}{t - c_k} \ c_k \in \mathbb{C}^*, \ n \in \mathbb{N},$$

which implies that $b_{\infty,0} - b_{0,0}$ is a non-positive integer.

If case (1) occurs and $b_{0,0} = 0$, we have $b_{\infty,0} - b_{0,0} = \alpha_0 \in \mathbb{Z}$. If case (1) occurs and $b_{0,0} \neq 0$, from Proposition 2.4, we see that $b_{0,0} = -\alpha_4$, because $x \equiv 0$. We then obtain

$$b_{\infty,0} - b_{0,0} = \alpha_0 - (-\alpha_4) = \alpha_0 + \alpha_4 = 1,$$

which is impossible.

If case (2) occurs, considering $b_{\infty,0} - b_{0,0} \in \mathbb{Z}$, we have $\alpha_0 \in \mathbb{Z}$. □

10.4.2 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$

Proposition 10.16. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that y, w both have a pole at $t = \infty$ and x, z are both holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. The proposition follows from Proposition 7.23. □

10.4.3 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$

Proposition 10.17. *Suppose that for $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, there exists a rational solution of type A such that y, w both have a pole at $t = \infty$ and x, z are both holomorphic at $t = \infty$. Moreover, assume that $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$. Then, $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.*

Proof. The proposition follows from Proposition 7.22. □

10.4.4 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$

Proposition 10.18. *Suppose that for* $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, *there exists a rational solution of type A such that* y, w *both have a pole at* $t = \infty$ *and* x, z *are both holomorphic at* $t = \infty$. *Moreover, assume that* $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$. *Then,* $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.

Proof. By Proposition 1.38, we first note that $a_{\infty,-1} = c_{\infty,-1} = \alpha_4$ and $b_{\infty,0} = -\alpha_0, d_{\infty,0} = \alpha_0$. Moreover, we may assume that $\alpha_0 \neq 0, \alpha_4 \neq 0$.

If x is holomorphic at $t = 0$, considering $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$, we have $\alpha_0 \in \mathbb{Z}$. If $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5$, considering $(b_{\infty,0} + d_{\infty,0}) - (b_{0,0} + d_{0,0}) \in \mathbb{Z}$, we obtain $\alpha_4 \in \mathbb{Z}$. Therefore, we have only to consider the following cases:
(1) x has a pole at $t = 0$ and y, z, w are all holomorphic at $t = 0$,
(2) x, z both have a pole at $t = 0$ and y, w are both holomorphic at $t = 0$.

From the discussions in Section 4, it follows that $h_{\infty,0} - h_{0,0} = 0$. On the other hand, by direct calculation, we have

$$a_{\infty,-1} - a_{0,-1} = \alpha_4 - (-\alpha_0 + \alpha_1) = \alpha_0 + \alpha_4 = 1,$$

which contradicts Lemma 10.1. □

10.5 Summary

Proposition 10.19. *Suppose that for* $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, *there exists a rational solution of type A. Then,* $\alpha_0 \in \mathbb{Z}, \alpha_4 \in \mathbb{Z}$.

Corollary 10.20. *Suppose that for* $D_5^{(1)}(\alpha_0, 0, 0, 0, \alpha_4, 0)$, *there exists a rational solution of type A. By some Bäcklund transformations, the parameters can then be transformed so that*

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 0, 0, 0, 1, 0).$$

11 Rational solutions of type A of $D_5^{(1)}(0, 0, 0, 0, 1, 0)$

11.1 The case where x, y, z, w are all holomorphic at $t = \infty$

11.1.1 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$

Proposition 11.1. *Suppose that for* $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, *there exists a rational solution such that* x, y, z, w *are all holomorphic at* $t = \infty$ *and* $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. *Then,* $(x, y, z, w) = (0, 0, 0, 0)$.

Proof. The proposition follows from the direct calculation and Proposition 1.2. □

11.1.2 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$

Proposition 11.2. *For $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, there exists no rational solution such that x, y, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$.*

Proof. By Propositions 1.1, 1.2 and their corollary, we first note that $y \equiv w \equiv 0$ and $a_{\infty,-1} = 0, c_{\infty,-1} = 1$. Since $-\alpha_0 + \alpha_1 = 0$, we next observe that x is holomorphic at $t = 0$.

From the discussions in Section 4, we find that $h_{\infty,0} - h_{0,0} = 0$, which implies that $x \equiv 0$ from Lemma 10.2. It then follows from the discussions in Section 2 and 3 that z is holomorphic at $t = 0$ and z has a pole of order at most one at $t = c \in \mathbb{C}^*$ with $\mathrm{Res}_{t=c} z = -1$. However, this contradicts the residue theorem. □

11.1.3 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$

Proposition 11.3. *For $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, there exists no rational solution such that x, y, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$.*

Proof. Suppose that $D_5^{(1)}(0, 0, 0, 0, 1, 0)$ has such a solution. By Propositions 1.1, 1.2 and their corollary, we first note that $y \equiv w \equiv 0$ and $a_{\infty,-1} = 1, c_{\infty,-1} = 0$. Since $-\alpha_0 + \alpha_1 = 0$, we next observe that x is holomorphic at $t = 0$.

From the discussions in Section 4, we have $h_{\infty,0} - h_{0,0} = 0$, which contradicts Lemma 10.1. □

11.1.4 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$

Proposition 11.4. *For $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, there exists no rational solution such that x, y, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$.*

Proof. It can be proved in the same way as Proposition 11.3. □

11.2 The case where y has a pole at $t = \infty$

11.2.1 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$

Proposition 11.5. *Suppose that for $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, there exists a rational solution such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. Then, $(x, y, z, w) = (0, -t, 0, 0)$.*

Proof. The proposition follows from the direct calculation and Propositions 1.7 and 1.8 □

11.2.2 The case where $(a_{\infty,0}, c_{\infty,0}) = (0,1)$

Proposition 11.6. *For $D_5^{(1)}(0,0,0,0,1,0)$, there exists no rational solution such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0,1)$.*

Proof. Suppose that $D_5^{(1)}(0,0,0,0,1,0)$ has such a solution. By Propositions 1.7 and 1.8 and their corollary, we first note that $y \equiv -1, w \equiv 0$ and $a_{\infty,-1} = 0, c_{\infty,-1} = -1$. Since $-\alpha_0 + \alpha_1 = 0$, we next observe that x is holomorphic at $t = 0$.

From the discussions in Section 4, we see that $h_{\infty,0} - h_{0,0} = 0$, which implies that $x \equiv 0$ from Lemma 10.2. From the discussions in Section 2, we then find that z is holomorphic at $t = 0$. Moreover, from the discussions in Section 3, we see that z is holomorphic at $t = c \in \mathbb{C}^*$ or z has a pole at $t = c$ with $\mathrm{Res}_{t=c} z = 1$. However, this contradicts the residue theorem. □

11.2.3 The case where $(a_{\infty,0}, c_{\infty,0}) = (1,0)$

Proposition 11.7. *For $D_5^{(1)}(0,0,0,0,1,0)$, there exists no rational solution such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1,0)$.*

Proof. Suppose that $D_5^{(1)}(0,0,0,0,1,0)$ has such a solution. From the discussions in Section 4, we have $h_{\infty,0} - h_{0,0} = -1$, which contradicts Proposition 4.22. □

11.2.4 The case where $(a_{\infty,0}, c_{\infty,0}) = (1,1)$

Proposition 11.8. *For $D_5^{(1)}(0,0,0,0,1,0)$, there exists no rational solution such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1,1)$.*

Proof. It can be proved in the same way as Proposition 11.7. □

11.3 The case where w has a pole at $t = \infty$

11.3.1 The case where $(a_{\infty,0}, c_{\infty,0}) = (0,0)$

Proposition 11.9. *Suppose that for $D_5^{(1)}(0,0,0,0,1,0)$, there exists a rational solution of type A such that w has a pole at $t=\infty$ and x,y,z are all holomorphic at $t=\infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0,0)$. Then, $(x,y,z,w) = (0,0,0,-t)$.*

Proof. The proposition follows from direct calculation and Proposition 1.24. □

11.3.2 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$

Proposition 11.10. *For $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, there exists no rational solution such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$.*

Proof. It can be proved in the same way as Proposition 11.7. □

11.3.3 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$

Proposition 11.11. *For $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, there exists no rational solution such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$.*

Proof. It can be proved in the same way as Proposition 11.3. □

11.3.4 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$

Proposition 11.12. *For $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, there exists no rational solution such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$.*

Proof. It can be proved in the same way as Proposition 11.7. □

11.4 The case where y, w have a pole at $t = \infty$

11.4.1 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$

Proposition 11.13. *Suppose that for $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, there exists a rational solution of type A such that y, w both have a pole at $t = \infty$ and x, z are both holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 0)$. Then, $(x, y, z, w) = (0, -t, 0, t)$.*

Proof. The proposition follows from direct calculation and Proposition 1.39. □

11.4.2 The case where $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$

Proposition 11.14. *For $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, there exists a rational solution of type A such that y, w both have a pole at $t = \infty$ and x, z are both holomorphic at $t = \infty$ and $(a_{\infty,0}, c_{\infty,0}) = (0, 1)$.*

Proof. It can be proved in the same way as Proposition 11.7. □

11.4.3 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$

Proposition 11.15. *For* $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, *there exists no rational solution of type A such that* y, w *both have a pole at* $t = \infty$ *and* x, z *are both holomorphic at* $t = \infty$ *and* $(a_{\infty,0}, c_{\infty,0}) = (1, 0)$.

Proof. It can be proved in the same way as Proposition 11.7. $\square$

11.4.4 The case where $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$

Proposition 11.16. *For* $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, *there exists no rational solution of type A such that* y, w *both have a pole at* $t = \infty$ *and* x, z *are both holomorphic at* $t = \infty$ *and* $(a_{\infty,0}, c_{\infty,0}) = (1, 1)$.

Proof. It can be proved in the same way as Proposition 11.3. $\square$

11.5 Summary

Proposition 11.17. *For* $D_5^{(1)}(0, 0, 0, 0, 1, 0)$, *there exists a rational solution of type A. Then,*

$$(x, y, z, w) = (0, 0, 0, 0), (0, -t, 0, 0), (0, 0, 0, -t), (0, -t, 0, t).$$

12 The main theorem for type A

We summarize the discussions in Sections 7, 9, 10 and 11.

Theorem 12.1. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, *there exists a rational solution of type A. By some Bäcklund transformations, the parameters and solution can then be transformed so that either of the following occurs:*

(1) $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_0, 0, 0, 0, \alpha_4, 0)$ *and*

$$(x, y, z, w) = (0, 0, 0, 0),$$

(2) $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 0, 0, 0, 1, 0)$ *and*

$$(x, y, z, w) = (0, 0, 0, 0), (0, -t, 0, 0), (0, 0, 0, -t), (0, -t, 0, t).$$

Moreover, $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has a rational solution of type A if and only if the parameters satisfy one of the following conditions:

(1)	$\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$	(2)	$\alpha_0 \in \mathbb{Z},\ \alpha_2+\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$
(3)	$\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$	(4)	$\alpha_0 \in \mathbb{Z},\ \alpha_2+\alpha_1 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$
(5)	$\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$	(6)	$\alpha_1 \in \mathbb{Z},\ \alpha_2+\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$
(7)	$\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_4 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$	(8)	$\alpha_1 \in \mathbb{Z},\ \alpha_2+\alpha_0 \in \mathbb{Z},\ \alpha_4+\alpha_3 \in \mathbb{Z},\ \alpha_5 \in \mathbb{Z},$
(9)	$\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$	(10)	$\alpha_0 \in \mathbb{Z},\ \alpha_2+\alpha_1 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$
(11)	$\alpha_0 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$	(12)	$\alpha_0 \in \mathbb{Z},\ \alpha_2+\alpha_1 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$
(13)	$\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$	(14)	$\alpha_1 \in \mathbb{Z},\ \alpha_2+\alpha_0 \in \mathbb{Z},\ \alpha_3 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$
(15)	$\alpha_1 \in \mathbb{Z},\ \alpha_2 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z},$	(16)	$\alpha_1 \in \mathbb{Z},\ \alpha_2+\alpha_0 \in \mathbb{Z},\ \alpha_3+\alpha_5 \in \mathbb{Z},\ \alpha_4 \in \mathbb{Z}.$

13 Rational solutions for the standard form I (1)

In this section, we treat the case where $-\alpha_0+\alpha_1=-\alpha_4+\alpha_5=0$.

Proposition 13.1. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, the parameters satisfy $-\alpha_0+\alpha_1=-\alpha_4+\alpha_5=0$. Then, by some shift operators, the parameters can be transformed so that $-\alpha_0+\alpha_1=-\alpha_4+\alpha_5=0$ and*

$$\begin{aligned}&\alpha_0,\ \alpha_2,\ \alpha_3,\ \alpha_4 \neq 0,\\ &\alpha_0+\alpha_1+2\alpha_2,\ 2\alpha_3+\alpha_4+\alpha_5 \neq 0,\\ &\alpha_0+\alpha_1+2\alpha_2+2\alpha_3,\ 2\alpha_2+2\alpha_3+\alpha_4+\alpha_5 \neq 0,\end{aligned}$$

which implies that

$$\alpha_0+\alpha_2,\ \alpha_4+\alpha_3,\ \alpha_0+\alpha_1,\ \alpha_4+\alpha_5 \neq 0.$$

Proposition 13.2. *Suppose that $-\alpha_0+\alpha_1=-\alpha_4+\alpha_5=0$ and $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has a rational solution of type B. Moreover, assume that the parameters satisfy not only the generic conditions in Proposition 13.1, but also*

$$2\alpha_3+\alpha_4+\alpha_5 \notin \mathbb{Z}, \alpha_0+\alpha_1 \notin \mathbb{Z}, \alpha_4+\alpha_5 \notin \mathbb{Z}.$$

By some Bäcklund transformations, the parameters and solution can then be transformed so that $-\alpha_0+\alpha_1=-\alpha_4+\alpha_5=0$, and

$$x=\frac{1}{2},\ y=-\frac{t}{2},\ z=\frac{1}{2},\ w=0.$$

Let us treat the case where for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=-\alpha_4+\alpha_5=0$ and one of $2\alpha_3+\alpha_4+\alpha_5$, $\alpha_0+\alpha_1$ and $\alpha_4+\alpha_5$ is an integer.

Proposition 13.3. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=-\alpha_4+\alpha_5=0$ and one of $2\alpha_3+\alpha_4+\alpha_5$, $\alpha_0+\alpha_1$ and $\alpha_4+\alpha_5$ is an integer. By some Bäcklund transformations, the parameters can then be transformed so that $\alpha_0=\alpha_1=1/2$ and $-\alpha_4+\alpha_5=0$.*

14 Rational solutions for the standard form I (2)

In this section, we treat the case where $\alpha_0=\alpha_1=1/2$ and $-\alpha_4+\alpha_5=0$. By shift operators, we can assume that $\alpha_2,\alpha_3,\alpha_4,\alpha_5\neq 0$.

Lemma 14.1. *Suppose that $\alpha_0=\alpha_1=1/2, -\alpha_4+\alpha_5=0$ and $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has a rational solution such that y,w both have a pole order n ($n\geq 2$) at $t=\infty$ and x,z are both holomorphic at $t=\infty$. Moreover, assume that $\alpha_4,\alpha_5,2\alpha_3+\alpha_4+\alpha_5\neq 0$. Then, $\alpha_2\in\mathbb{Z}$.*

Proof. From the discussion about the meromorphic solutions at $t=0$ in Section 2, we have only to consider the following three cases:
(0) x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}=0$,
(1) z has a pole at $t=0$ and x,y,w are all holomorphic at $t=0$,
(2) y,w both have a pole at $t=0$ and x,z are both holomorphic at $t=0$.

We treat case (1). If case (0) or (2) occurs, the proposition can be proved in the same way.

From Propositions 4.9, 4.12 and 4.22, we have

$$h_{\infty,0}-h_{0,0}=\alpha_2^2-\alpha_3(\alpha_3+\alpha_4+\alpha_5)\in\mathbb{Z}.$$

Since $\alpha_3\neq 0$, $s_3(x,y,z,w)$ is a solution of $D_5^{(1)}(1/2,1/2,\alpha_2+\alpha_3,-\alpha_3,\alpha_4+\alpha_3,\alpha_5+\alpha_3)$ such that both of $s_3(y,w)$ have a pole of order n at $t=\infty$ and both of $s_3(x,z)$ are holomorphic at $t=\infty$. Moreover, all of $s_3(x,y,z,w)$ are holomorphic at $t=0$ and $b_{0,0}=d_{0,0}=0$. By Proposition 4.9, 4.10 and 4.22, we obtain

$$h_{\infty,0}-h_{0,0}=-(\alpha_2+\alpha_3)(\alpha_2+\alpha_3+\alpha_4+\alpha_5)\in\mathbb{Z}.$$

Thus, it follows that

$$\{\alpha_2^2-\alpha_3(\alpha_3+\alpha_4+\alpha_5)\}-\{-(\alpha_2+\alpha_3)(\alpha_2+\alpha_3+\alpha_4+\alpha_5)\}=\alpha_2(2\alpha_2+2\alpha_3+\alpha_4+\alpha_5)=\alpha_2\in\mathbb{Z},$$

because $\alpha_0=\alpha_1=1/2$. $\square$

Proposition 14.2. *Suppose that $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has a rational solution of type B and the parameters satisfy*

$$\alpha_0 = \alpha_1 = 1/2,\ -\alpha_4 + \alpha_5 = 0, 2\alpha_3 + \alpha_4 + \alpha_5 \notin \mathbb{Z},\ \alpha_4 + \alpha_5 \notin \mathbb{Z}.$$

By some Bäcklund transformations, the parameters and solution can then be transformed so that $-\alpha_0 + \alpha_1 = -\alpha_4 + \alpha_5 = 0$ and $x = 1/2$, $y = -t/2$, $z = 1/2$, $w = 0$.

Let us treat the case where $\alpha_0 = \alpha_1 = 1/2$, $-\alpha_4 + \alpha_5 = 0$ and either $2\alpha_3 + \alpha_4 + \alpha_5$ or $\alpha_4 + \alpha_5$ is an integer.

Proposition 14.3. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $\alpha_0 = \alpha_1 = 1/2$, $-\alpha_4; \alpha_5 = 0$ and either $2\alpha_3 + \alpha_4 + \alpha_5$ or $\alpha_4 + \alpha_5$ is an integer. By some Bäcklund transformations, the parameters can then be transformed so that either of the following occurs:*
(1) $\alpha_0 = \alpha_1 = 1/2$ *and* $\alpha_4 = \alpha_5 = 0$, (2) $\alpha_0 = \alpha_1 = \alpha_4 = \alpha_5 = 1/2$.

15 Rational solutions for the standard form I (3)

In this section, we treat the case where $\alpha_0 = \alpha_1 = 1/2$ and $\alpha_4 = \alpha_5 = 0$, and the case where $\alpha_0 = \alpha_1 = \alpha_4 = \alpha_5 = 1/2$. For this purpose, we have the following lemma:

Lemma 15.1. *Suppose that $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has a rational solution of type B such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$. Moreover, assume that $-\alpha_0 + \alpha_1 = 0$ and $h_{\infty,0} - h_{0,0} = 0$. Then, $x \equiv 1/2$.*

Proof. The lemma follows from Lemma 10.2. □

15.1 The case where $\alpha_0 = \alpha_1 = 1/2$ and $\alpha_4 = \alpha_5 = 0$

Proposition 15.2. *Suppose that $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has a rational solution of type B and the parameters satisfy*

$$\alpha_0 = \alpha_1 = 1/2,\ \alpha_4 = \alpha_5 = 0 \text{ and } 2\alpha_2, 2\alpha_3 \notin \mathbb{Z}.$$

By some Bäcklund transformations, the parameters and solution can then be transformed so that either of the following occurs:
(1) $-\alpha_0 + \alpha_1 = -\alpha_4 + \alpha_5 = 0$ *and* $(x, y, z, w) = (1/2, -t/2, 1/2, 0)$,
(2) $\alpha_0 = \alpha_1 = 1/2$, $-\alpha_4 + \alpha_5 = 2\alpha_3 + \alpha_4 + \alpha_5 = 0$, *and*

$$(x, y, z, w) = (1/2, -t/2 + b, 1/2, d),$$

where b, d are both arbitrary complex numbers and satisfy $b + d = 0$.

Proof. From the discussion in Section 1, we consider the following cases:
(1) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$,
(2) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$,
(3) y,z both have a pole at $t=\infty$ and x,w are both holomorphic at $t=\infty$,
(4) y,w both have a pole of order n ($n\geq 2$) at $t=\infty$ and x,z are both holomorphic at $t=\infty$.

If case (1) or (2) occurs, the proposition follows from Propositions 1.11 and 1.17. We treat case (3). If case (4) occurs, the proposition can be proved in the same way.

Since $2\alpha_3\notin\mathbb{Z}$, it follows from Proposition 1.32 that $s_3(x,y,z,w)$ is a rational solution of $D_5^{(1)}(1/2,1/2,0,-\alpha_3,\alpha_3,\alpha_3)$ such that $s_3(y)$ has a pole at $t=\infty$ and all of $s_3(x,z,w)$ are holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$. Therefore, the proposition follows from 1.12. □

The remaining case is

$$\alpha_0=\alpha_1=1/2,\ \alpha_4=\alpha_5=0,\ 2\alpha_2\in\mathbb{Z},2\alpha_3\in\mathbb{Z}.$$

Proposition 15.3. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $\alpha_0=\alpha_1=1/2$, $\alpha_4=\alpha_5=0$ and $2\alpha_2\in\mathbb{Z},2\alpha_3\in\mathbb{Z}$. By some Bäcklund transformations, the parameters can then be transformed that*

$$(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=(1/2,1/2,0,0,0,0),(0,0,1/2,0,0,0).$$

15.2 The case where $\alpha_0=\alpha_1=\alpha_4=\alpha_5=1/2$

Proposition 15.4. *Suppose that $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$. Moreover, assume that $\alpha_0=\alpha_1=1/2$, $\alpha_2=-1/2$, $-\alpha_4+\alpha_5=0$ and $h_{\infty,0}-h_{0,0}=0$. One of the following then occurs:*
(1) $-\alpha_0+\alpha_1=-\alpha_4+\alpha_5=0$ *and*

$$x=\frac{1}{2},\ y=-\frac{t}{2},\ z=\frac{1}{2},\ w=0,$$

(2) $\alpha_0=\alpha_1=1/2,\ \alpha_2=-1/2,\ \alpha_3=1/2,\ \alpha_4=\alpha_5=0$ *and*

$$x=\frac{1}{2},\ y=-\frac{t}{2}+\frac{2\alpha_3}{2c-1},\ z=c,\ w=-\frac{2\alpha_3}{2c-1},$$

(3) $\alpha_0=\alpha_1=1/2,\ \alpha_2=-1/2,\ \alpha_3=0,\ \alpha_4=\alpha_5=1/2$ *and*

$$x=\frac{1}{2},\ y=-\frac{t}{2},\ z=\frac{1}{2}+Ct^{-1},\ w=0,$$

(4) $\alpha_0=\alpha_1=1/2$, $\alpha_2=-1/2$, $\alpha_3=1$, $\alpha_4=\alpha_5=-1/2$ *and*

$$x=\frac{1}{2},\ y=-\frac{t}{2},\ z=\frac{1}{2}+Ct^{-1},\ w=\frac{Ct}{(t/2+C)(t/2-C)},$$

where c, C are both arbitrary complex numbers and $c\neq 1/2$.

Proof. If $c_{\infty,0}\neq 1/2$, it follows from Propositions 1.16 and 1.17 that $\alpha_4=\alpha_5=0$ and

$$x=\frac{1}{2},\ y=-\frac{t}{2}+\frac{2\alpha_3}{2c-1},\ z=c,\ w=-\frac{2\alpha_3}{2c-1}.$$

Therefore, we may assume that $c_{\infty,0}=1/2$. Moreover, by Lemma 15.1, we note that $x\equiv 1/2$. Substituting $x=1/2$ in

$$ty'=-2xy^2+y^2-2txy+\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}\,y-\alpha_1 t,$$

we obtain $y=-t/2+b$, where b is an arbitrary constant. Proposition 1.10 shows that $b_{\infty,0}=0$, which implies that $y=-t/2$.

Substituting $x=1/2$ and $y=-t/2$ in

$$\begin{aligned}tx'=2x^2y+tx^2-2xy-\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}\,x\\ +(\alpha_2+\alpha_5)+2z\{(z-1)w+\alpha_3\},\end{aligned}$$

we have

$$2z(z-1)w=-\alpha_5+\frac{1}{2}-2\alpha_3 z.$$

Substituting $x=1/2$ and $y=-t/2$ in

$$tz'=2z^2w+tz^2-2zw-\{t+(\alpha_5+\alpha_4)\}\,z+\alpha_5+2yz(z-1),$$

we have $tz'=-z+1/2$, because $2z(z-1)w=-\alpha_5+1/2-2\alpha_3 z$. By solving this differential equation, we obtain $z=1/2+Ct^{-1}$, where C is an arbitrary constant.

If $C=0$, it follows from Proposition 1.13 that

$$x=\frac{1}{2},\ y=-\frac{t}{2},\ z=\frac{1}{2},\ w=0.$$

We assume that $C\neq 0$. Considering

$$x=\frac{1}{2},\ y=-\frac{t}{2},\ z=\frac{1}{2}+Ct^{-1},\ 2z(z-1)w=-\alpha_5+\frac{1}{2}-2\alpha_3 z,$$

we have

$$w=\frac{\alpha_3 Ct}{(t/2+C)(t/2-C)}.$$

Substituting this solution in

$$tw' = -2zw^2 + w^2 - 2tzw + \{t + (\alpha_5 + \alpha_4)\}\, w - \alpha_3 t - 2y(-w + 2zw + \alpha_3),$$

we then have

$$\frac{\alpha_3 Ct}{\{(t/2+C)(t/2-C)\}^2}\left(-\frac{t^2}{4} - C^2\right) = \frac{\alpha_3 Ct}{\{(t/2+C)(t/2-C)\}^2}\left(\frac{t^2}{4}(1-2\alpha_3) - C^2\right),$$

which implies that $\alpha_3 = 0, 1$. □

Proposition 15.5. *Suppose that $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ has a rational solution of type B and the parameters satisfy*

$$\alpha_0 = \alpha_1 = \alpha_4 = \alpha_5 = 1/2, \ \textit{and}\ 2\alpha_2, 2\alpha_3 \notin \mathbb{Z}.$$

By some Bäcklund transformations, the parameters and solution can then be transformed so that $-\alpha_0 + \alpha_1 = -\alpha_4 + \alpha_5 = 0$ and $(x, y, z, w) = (1/2, -t/2, 1/2, 0)$.

Proof. By the discussion in Section 1, we consider the following cases:
(1) y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$,
(2) y, z both have a pole at $t = \infty$ and x, w are both holomorphic at $t = \infty$,
(3) y, w both have a pole at $t = \infty$ and x, z are both holomorphic at $t = \infty$.

We treat case (2). If case (1) or (3) occurs, the proposition can be proved in the same way. For this purpose, we consider the following cases:
(i) x, y, z, w are all holomorphic at $t = 0$,
(ii) z has a pole at $t = 0$ and x, y, w are holomorphic at $t = 0$,
(iii) y, w both have a pole at $t = 0$ and x, z are both holomorphic at $t = 0$.

We deal with case (ii). If case (i) or (iii) occurs, the proposition can be proved in the same way. Since $2\alpha_3 \notin \mathbb{Z}$, it follows from Propositions 1.32 and 2.11 that $s_3(x, y, z, w)$ is a rational solution of $D_5^{(1)}(1/2, 1/2, -1/2, -\alpha_3, \alpha_3+1/2, \alpha_3+1/2)$ such that $s_3(y)$ has a pole at $t = \infty$ and all of $s_3(x, z, w)$ are holomorphic at $t = \infty$. Moreover, all of $s_3(x, y, z, w)$ are holomorphic at $t = 0$ and $b_{0,0} = d_{0,0} = 0$. Therefore, from Propositions 4.5 and 4.10, we see that for $s_3(x, y, z, w)$, $h_{\infty,0} - h_{0,0} = 0$. Thus, the proposition follows from Proposition 15.4. □

The remaining case is

$$\alpha_0 = \alpha_1 = \alpha_4 = \alpha_5 = 1/2,\ 2\alpha_2 \in \mathbb{Z}, 2\alpha_3 \in \mathbb{Z}.$$

Proposition 15.6. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$,*

$$\alpha_0 = \alpha_1 = \alpha_4 = \alpha_5 = 1/2,\ 2\alpha_2 \in \mathbb{Z},\ 2\alpha_3 \in \mathbb{Z}.$$

By some Bäcklund transformations, the parameters can then be transformed that

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 0, 1/2, 0, 0, 0).$$

16 Rational solutions for the standard form I (4)

16.1 Rational solutions of $D_5^{(1)}(1/2, 1/2, 0, 0, 0, 0)$

Lemma 16.1. *Suppose that $D_5^{(1)}(1/2, 1/2, -1, 1, 0, 0)$ has a rational solution of type B such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} = 1/2$. Moreover, assume that $h_{\infty,0} - h_{0,0} = 0$. Then, $(x, y, z, w) = (1/2, -t/2, 1/2, 0)$.*

Proof. By Lemma 15.1, we note that $x \equiv 1/2$. Substituting $x = 1/2$ in

$$ty' = -2xy^2 + y^2 - 2txy + \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\} y - \alpha_1 t,$$

we obtain $y = -t/2 + b$, where b is an arbitrary constant. Proposition 1.10 shows that $b_{\infty,0} = 0$, which implies that $y = -t/2$.

Substituting $x \equiv 1/2$ and $y = -t/2$ in

$$\begin{aligned} tx' = 2x^2y + tx^2 - 2xy - \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\} x \\ + (\alpha_2 + \alpha_5) + 2z \{(z - 1)w + \alpha_3\}, \end{aligned}$$

we have $2z(z - 1)w = 1 - 2z$.

Substituting $x \equiv 1/2$ and $y = -t/2$ in

$$tz' = 2z^2w + tz^2 - 2zw - \{t + (\alpha_5 + \alpha_4)\} z + \alpha_5 + 2yz(z - 1),$$

we have $tz' = 1 - 2z$, because $2z(z - 1)w = 1 - 2z$. By solving this differential equation, we obtain $z = 1/2 + Ct^{-2}$, where C is an arbitrary constant.

Considering $2z(z - 1)w = 1 - 2z$, we have

$$w = \frac{1 - 2z}{2z(z - 1)} = \frac{C}{(t^2/2 + C)(t^2/2 - C)} = 4Ct^{-4} + \cdots \quad \text{at } t = \infty.$$

If $C = 0$, the proposition follows. Therefore, we assume that $C \neq 0$.

Substituting $x \equiv 1/2$, $y = -t/2$ and $z = 1/2 + Ct^{-2}$ in

$$tw' = -2zw^2 + w^2 - 2tzw + \{t + (\alpha_5 + \alpha_4)\} w - \alpha_3 t - 2y(-w + 2zw + \alpha_3),$$

we obtain

$$tw' = -2Ct^{-2}w^2,$$

which implies that

$$w = -\frac{t^2}{C + Dt^2} = \begin{cases} -t^2/C \text{ if } D = 0, \\ -1/D + \cdots \text{ at } t = \infty \text{ if } D \neq 0, \end{cases}$$

where D is an arbitrary constant. However, this is impossible. Thus, it follows that $C = 0$, which proves the lemma. □

Lemma 16.2. *Suppose that $D_5^{(1)}(1/2,1/2,-1,1,0,0)$ has a rational solution of type B. such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$. Then, $(x,y,z,w)=(1/2,-t/2,1/2,0)$.*

Proof. By Proposition 4.5, we note that $h_{\infty,0}=0$. From the discussion in Section 1, let us consider the following cases:
(1) x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}=0$,
(2) x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}\neq 0$,
(3) z has a pole at $t=0$ and x,y,w are all holomorphic at $t=0$,
(4) y,w both have a pole at $t=0$ and x,z are both holomorphic at $t=0$.

From the discussion in Section 4, $h_{0,0}=0,1/4,1/4,1/4$ if case (1), (2), (3) or (4) occurs. Since $h_{\infty,0}-h_{0,0}\in\mathbb{Z}$, it follows that $h_{\infty,0}-h_{0,0}=0$, which proves the lemma. □

Proposition 16.3. *Suppose that $D_5^{(1)}(1/2,1/2,0,0,0,0)$ has a rational solution of type B. The parameters and solution can then be transformed so that one of the following occurs:*
(1) $-\alpha_0+\alpha_1=-\alpha_4+\alpha_5=0$, *and* $(x,y,z,w)=(1/2,-t/2,1/2,0)$,
(2) $\alpha_0=\alpha_1=1/2$, $-\alpha_4+\alpha_5=2\alpha_3+\alpha_4+\alpha_5=0$, *and*

$$(x,y,z,w)=(1/2,-t/2+b,1/2,d),$$

where b,d are both arbitrary complex numbers and satisfy $b+d=0$.

Proof. For the proof, we have only to consider the following:
(1) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$,
(2) y,z both have a pole at $t=\infty$ and x,w are both holomorphic at $t=\infty$,
(3) y,w both have a pole of order n ($n\geq 2$) at $t=\infty$ and x,z are both holomorphic at $t=\infty$.

If case (1) occurs, the proposition follows from the direct calculations, Propositions 1.12 and 1.17.

If case (2) occurs, $s_3s_2s_1s_0(x,y,z,w)$ is a rational solution of $D_5^{(1)}(1/2,1/2,0,-1,1,1)$ such that $s_3s_2s_1s_0(y)$ has a pole at $t=\infty$ and all of $s_3s_2s_1s_0(x,z,w)$ are holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$. Therefore, the proposition follows from Proposition 1.12.

If case (3) occurs, $s_2s_1s_0(x,y,z,w)$ is a rational solution of $D_5^{(1)}(1/2,1/2,-1,1,0,0)$ such that $s_2s_1s_0(y)$ has a pole at $t=\infty$ and all of $s_2s_1s_0(x,z,w)$ are holomorphic at $t=\infty$. If $c_{\infty,0}\neq 1/2$, by s_3 and Proposition 1.17, we can obtain the proposition. If $c_{\infty,0}=1/2$, by Lemma 16.2, we can prove the proposition. □

16.2 Rational solutions of $D_5^{(1)}(0,0,1/2,0,0,0)$

Lemma 16.4. *Suppose that $D_5^{(1)}(0,0,1/2,0,0,0)$ has a rational solution of type B. Then, $h_{\infty,0}-h_{0,0}=0$.*

Proof. From the discussion in Section 1, we first consider the following:
(1) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$,
(2) w has a pole at $t=\infty$ and x,y,z are all holomorphic at $t=\infty$,
(3) y,z both have a pole at $t=\infty$ and x,w are both holomorphic at $t=\infty$,
(4) y,w both have a pole at $t=\infty$ and x,z are both holomorphic at $t=\infty$.
From the discussion in Section 1, it then follows that $h_{\infty,0}=0,-1/4,0,-1/4$ if case (1), (2), (3) or (4) occurs.

From the discussion in Section 2, we next consider the following:
(i) x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}=0$,
(ii) x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}\neq 0$,
(iii) z has a pole at $t=0$ and x,y,w are all holomorphic at $t=0$,
(iv) y,w both have a pole at $t=0$ and x,z are both holomorphic at $t=0$.
From the discussion in Section 2, it then follows that $h_{0,0}=0,-1/4,0,-1/4$ if if case (i), (ii), (iii) or (iv) occurs.

Since $h_{\infty,0}-h_{0,0}=0$, we find that $h_{\infty,0}-h_{0,0}=0$. □

Lemma 16.5. *Suppose that $D_5^{(1)}(0,0,1/2,0,0,0)$ has a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$. Then, $(x,y,z,w)=(1/2,-t/2,1/2,0)$.*

Proof. By Lemmas 15.1 and 16.4, we note that $x\equiv 1/2$. Substituting $x=1/2$ in

$$ty'=-2xy^2+y^2-2txy+\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}\,y-\alpha_1 t,$$

we obtain $y=Ct$ where C is an arbitrary constant. Proposition 1.10 shows that $C=-1/2$, which implies that $y=-t/2$.

Substituting $x\equiv 1/2$ and $y=-t/2$ in

$$\begin{aligned}tx'=2x^2y+tx^2-2xy-\{t+(2\alpha_2+2\alpha_3+\alpha_5+\alpha_4)\}\,x\\ +(\alpha_2+\alpha_5)+2z\,\{(z-1)w+\alpha_3\}\,,\end{aligned}$$

we have $2z(z-1)w=0$.

Substituting $x=1/2$ and $y=-t/2$ in

$$tz'=2z^2w+tz^2-2zw-\{t+(\alpha_5+\alpha_4)\}\,z+\alpha_5+2yz(z-1),$$

we have $tz'=0$, because $2z(z-1)w=0$. Therefore, from Proposition 1.10, we obtain $z\equiv 1/2$, which implies that $w\equiv 0$. □

Lemma 16.6. *Suppose that $D_5^{(1)}(0,0,0,1/2,0,0)$ has a rational solution of type B. Then, $h_{\infty,0}-h_{0,0}=0$.*

Proof. It can be proved in the same way as Lemma 16.4. $\square$

Lemma 16.7. *$D_5^{(1)}(0,0,0,1/2,0,0)$ has no rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$.*

Proof. Suppose that $D_5^{(1)}(0,0,0,1/2,0,0)$ has such a solution. By Lemmas 15.1 and 16.6, we then note that $x\equiv 1/2$. Substituting $x=1/2$ in

$$ty' = -2xy^2 + y^2 - 2txy + \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\}\, y - \alpha_1 t,$$

we obtain $y = Ct$ where C is an arbitrary constant. Proposition 1.16 shows that $C = -1/2$, which implies that $y = -t/2$.

Substituting $x\equiv 1/2$ and $y=-t/2$ in

$$\begin{aligned} tx' = 2x^2y + tx^2 - 2xy - \{t + (2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4)\}\, x \\ + (\alpha_2 + \alpha_5) + 2z\,\{(z-1)w + \alpha_3\}\,, \end{aligned}$$

we have $2z(z-1)w = 1/2 - z$.

Substituting $x=1/2$ and $y=-t/2$ in

$$tz' = 2z^2w + tz^2 - 2zw - \{t + (\alpha_5 + \alpha_4)\}\, z + \alpha_5 + 2yz(z-1),$$

we have $tz' = 1/2 - z$, because $2z(z-1)w = 1/2 - z$. By this differential equation, we obtain $z = 1/2 + Ct^{-1}$ $C\in\mathbb{C}$, which is impossible. $\square$

Proposition 16.8. *Suppose that $D_5^{(1)}(0,0,1/2,0,0,0)$ has a rational solution of type B. Then, $(x,y,z,w) = (1/2,-t/2,1/2,0)$.*

Proof. From the discussion in Section 1, we consider the following:
(1) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$,
(2) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$,
(3) w has a pole at $t=\infty$ and x,y,z are all holomorphic at $t=\infty$,
(4) y,z both have a pole at $t=\infty$ and x,w are both holomorphic at $t=\infty$,
(5) y,w both have a pole of order $n\,(n\geq 1)$ at $t=\infty$ and x,z are both holomorphic at $t=\infty$.

If case (1), (2) or (4) occurs, since $h_{\infty,0}-h_{0,0}=0$, we first note that $x\equiv 1/2$. If case (1) occurs, we can prove the proposition by substituting $x\equiv 1/2$ in $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$. If case (2) or (4) occurs, we can show contradiction by substituting $x\equiv 1/2$ in $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$.

If case (3) occurs, by π_2, we find that $D_5^{(1)}(0,0,0,1/2,0,0)$ has a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$. However, this is impossible from Lemma 16.7.

If case (5) occurs and $n=1$, by π_2, we can show the contradiction. If case (5) occurs and $n\geq 2$, we can prove the contradiction by substituting $x\equiv 1/2$ in $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$. $\square$

17 Summary for the standard form I

We summarize the discussions in Sections 10, 11, 12 and 13.

Theorem 17.1. *Suppose that* $-\alpha_0+\alpha_1=-\alpha_4+\alpha_5=0$ *and* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ *has a rational solution type B. The parameters and solutions can then be transformed so that either of the following occurs:*
(1) $-\alpha_0+\alpha_1=-\alpha_4+\alpha_5=0$ *and* $(x,y,z,w)=(1/2,-t/2,1/2,0)$,
(2) $(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=(1/2,1/2,0,\alpha_3,-\alpha_3,-\alpha_3)$ *and*

$$(x,y,z,w)=(1/2,-t/2+b,1/2,d),$$

where b,d *are both arbitrary complex numbers and satisfy* $b+d=0$.

Remark

We note that if $\alpha_0=\alpha_1=1/2$ and $-\alpha_4+\alpha_5=2\alpha_3+\alpha_4+\alpha_5=0$, this means that $(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=(1/2,1/2,0,\alpha_3,-\alpha_3,-\alpha_3)$.

18 Rational solutions for the standard form II (1)

In this section, we treat the standard form II, the case where $-\alpha_0+\alpha_1=0$, $-\alpha_4+\alpha_5=1$. For this purpose, we obtain more necessary conditions, if $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has a rational solution of type B.

We then may assume that $2\alpha_3+\alpha_4+\alpha_5\neq 0$. Moreover, since $\mathrm{Res}_{t=0}x=-\alpha_0+\alpha_1=0$, we can assume that x is holomorphic at $t=0$.

18.1 The case where y has a pole at $t=\infty$ and $c_{\infty,0}=1/2\cdots$ (0)

In this subsection, we treat the case where y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$.

18.1.1 The case where x,y,z,w are all holomorphic at $t=0$

Proposition 18.1. *If* $-\alpha_0+\alpha_1=0$ *and* $-\alpha_4+\alpha_5=1$, $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ *has no rational solution of type B such that both* (1) *and* (2) *hold:*
(1) y *has a pole at* $t=\infty$ *and* x,z,w *are holomorphic at* $t=\infty$ *and* $c_{\infty,0}=1/2$,
(2) x,y,z,w *are all holomorphic at* $t=0$ *and* $b_{0,0}=d_{0,0}=0$.

Proof. Suppose that $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has such a rational solution. From Proposition 4.5, 4.10 and 4.22, it follows that

$$h_{\infty,0} - h_{0,0} = 1/4 \in \mathbb{Z},$$

which is impossible. □

Proposition 18.2. *Suppose that $-\alpha_0 + \alpha_1 = 0$, $-\alpha_4 + \alpha_5 = 1$ and $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has a rational solution of type B such that both* (1) *and* (2) *hold:*
(1) *y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} = 1/2$,*
(2) *x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = 0,\ b_{0,0} \neq 0$.*
Then, $\alpha_2 - 1/2 \in \mathbb{Z}$.

Proof. From Proposition 4.5, 4.10, it follows that

$$h_{\infty,0} - h_{0,0} = 1/4 + \alpha_2(\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5) \in \mathbb{Z}. \tag{18.1}$$

$\pi_2(x, y, z, w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0)$ such that $\pi_2(y)$ has a pole at $t = \infty$ and all of $\pi_2(x, z, w)$ are holomorphic at $t = \infty$ and $c_{\infty,0} = 1/2$. Moreover, $\pi_2(z)$ has a pole at $t = 0$ and all of $\pi_2(x, y, w)$ are holomorphic at $t = 0$. Then, for $\pi_2(x, y, z, w)$, we observe from Propositions 4.5 and 4.12 that

$$h_{\infty,0} - h_{0,0} = -1/4 - \alpha_2(\alpha_0 + \alpha_1 + \alpha_2) \in \mathbb{Z}. \tag{18.2}$$

Therefore, it follows from (18.1) and (18.2) that

$$-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1 \text{ and } \alpha_2 - 1/2 \in \mathbb{Z}.$$

□

Proposition 18.3. *Suppose that $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1$ and $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ has a rational solution of type B such that both* (1) *and* (2) *hold:*
(1) *y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} = 1/2$,*
(2) *x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5 = 1$.*
Either of the following then occurs: (i) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, (ii) $\alpha_4 + \alpha_5 \in \mathbb{Z}$.

Proof. If $b_{0,0} \neq 0$, it follows from the proof of Proposition 8.5 that $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1$ and $\alpha_0 + \alpha_1 \in \mathbb{Z}$.

Let us consider the case where $b_{0,0} = 0$, which implies that $d_{0,0} = -\alpha_4 + \alpha_5 = 1 \neq 0$. For this purpose, by Proposition 2.5, we have only to consider the following two cases: (i) $b_{0,0} = 0$ and $c_{0,0} = 1/2$, (ii) $b_{0,0} = 0$ and $c_{0,0} = \alpha_5/(-\alpha_4 + \alpha_5) \neq 1/2$.

If case (i) occurs, it follows from Proposition 2.6 that $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1$ and $\alpha_4 + \alpha_5 = 0$.

Let us treat case (ii). For this purpose, we can assume that $\alpha_4 \neq 0$, because $-\alpha_4+\alpha_5 = 1$. $s_1\pi_2(x,y,z,w)$ is then a rational solution of type B of $D_5^{(1)}(\alpha_5, -\alpha_4, \alpha_3+\alpha_4, \alpha_2, \alpha_1, \alpha_0)$ such that $s_1\pi_2(y)$ has a pole at $t=\infty$ and all of $s_1\pi_2(x,z,w)$ are holomorphic at $t=\infty$ and $c_{\infty,0} = 1/2$. Moreover, all of $s_1\pi_2(x,y,z,w)$ are holomorphic at $t=0$. Thus, it follows from the residue theorem that $-\alpha_0+\alpha_1 = 0,\ -\alpha_4+\alpha_5 = 1$ and

$$-\alpha_5 + (-\alpha_4) = -\mathrm{Res}_{t=\infty} s_1\pi_2(x) \in \mathbb{Z}.$$

□

18.1.2 The case where z has a pole at $t=0$

Proposition 18.4. *Suppose that $-\alpha_0+\alpha_1 = 0,\ -\alpha_4+\alpha_5 = 1$ and $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ has a rational solution of type B such that both* (1) *and* (2) *hold:*
(1) *y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0} = 1/2$,*
(2) *z has a pole of order n $(n \ge 1)$ at $t=0$ and x,y,w are all holomorphic at $t=0$.*
Then, one of the following occurs: (i) $\alpha_2 = 0$, (ii) $\alpha_0+\alpha_2 = 0$, (iii) $\alpha_3 - 1/2 \in \mathbb{Z}$.

Proof. We can assume that $\alpha_2 \neq 0$. From Proposition 4.5 and 4.12, it follows that

$$h_{\infty,0} - h_{0,0} = 1/4 - \alpha_3(\alpha_3+\alpha_4+\alpha_5) \in \mathbb{Z}. \tag{18.3}$$

$s_2(x,y,z,w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_0+\alpha_2, \alpha_1+\alpha_2, -\alpha_2, \alpha_3+\alpha_2, \alpha_4, \alpha_5)$ such that one of the following occurs:
(i) $s_2(w)$ has a pole at $t=\infty$ and all of $s_2(x,y,z)$ are holomorphic at $t=\infty$,
(ii) both of $s_2(y,w)$ have a pole of order one at $t=\infty$ and both of $s_2(x,z)$ are holomorphic at $t=\infty$,
(iii) both of $s_2(y,w)$ have a pole of order m $(m \ge 2)$ at $t=\infty$ and both of $s_2(x,z)$ are holomorphic at $t=\infty$ and $s_2(z)$ has a pole of order n $(n \ge 2)$ at $t=0$ and all of $s_2(x,y,w)$ are holomorphic at $t=0$
(iv) both of $s_2(y,w)$ have a pole of order m $(m \ge 2)$ at $t=\infty$ and both of $s_2(x,z)$ are holomorphic at $t=\infty$ and $s_2(z)$ has a pole of order one at $t=0$ and all of $s_2(x,y,w)$ are holomorphic at $t=0$.

If case (i) occurs, it follows from Proposition 1.26 that $-\alpha_0+\alpha_1 = 0,\ -\alpha_4+\alpha_5 = 1$ and $\alpha_1+\alpha_2 = 0$.

If case (ii) occurs, it follows from Proposition 1.41 that $-\alpha_0+\alpha_1 = 0,\ -\alpha_4+\alpha_5 = 1$ and $\alpha_1+\alpha_2 = 0$.

If case (iii) occurs, $\pi_2 s_2(x,y,z,w)$ is then a rational solution of $D_5^{(1)}(\alpha_5, \alpha_4, \alpha_3 + \alpha_2, -\alpha_2, \alpha_1+\alpha_2, \alpha_0+\alpha_2)$ such that $\pi_2 s_2(y)$ and $\pi_2 s_2(z)$ have a pole of order one and

$m-1$ at $t=\infty$, respectively and $\pi_2 s_2(y)$ and $\pi_2 s_2(w)$ have a pole of order $n-1$ at $t=0$. For $\pi_2 s_2(x,y,z,w)$, we observe from Proposition 4.8 and 4.14 that

$$h_{\infty,0} - h_{0,0} = -1/4 + \alpha_3(\alpha_0+\alpha_1+2\alpha_2+\alpha_3) \in \mathbb{Z}. \tag{18.4}$$

From (18.3) and (18.4), it follows that $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1$ and $\alpha_3-1/2\in\mathbb{Z}$.

If case (iv) occurs, we use $\pi_2 s_2$ in the same way and can prove that $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1$ and $\alpha_3-1/2\in\mathbb{Z}$. □

18.1.3 The case where y,w have a pole at $t=0$

Proposition 18.5. *Suppose that $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1$ and $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ has a rational solution of type B such that both (1) and (2) hold:*
(1) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$
(2) y,w both have a pole of order n $(n\ge 1)$ at $t=0$ and x,z are both holomorphic at $t=0$.
Either of the following then occurs: (i) $\alpha_2=0$, (ii) $\alpha_4+\alpha_5=0$.

Proof. We assume that $\alpha_2\neq 0$. From Corollary 2.19, it follows that $s_2(x,y,z,w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_0+\alpha_2,\alpha_1+\alpha_2,-\alpha_2,\alpha_3+\alpha_2,\alpha_4,\alpha_5)$ such that all of $s_2(x,y,z,w)$ are holomorphic at $t=0$. Moreover, the constant terms of $s_2(x,z)$ at $t=0$ are given by $a_{0,0}=c_{0,0}=1/2$ and for $s_2(y,w)$, $b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5$. Therefore, it follows from Lemma 2.20 that $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1$ and $\alpha_4+\alpha_5=0$. □

18.1.4 Summary

Proposition 18.6. *Suppose that $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1$ and $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$ has a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$. Then, one of the following occurs:*

(1) $2\alpha_3+\alpha_4+\alpha_5\in\mathbb{Z}$, (2) $\alpha_2=0$, (3) $\alpha_2-1/2\in\mathbb{Z}$, (4) $\alpha_3-1/2\in\mathbb{Z}$,
(5) $\alpha_0+\alpha_1\in\mathbb{Z}$, (6) $\alpha_4+\alpha_5\in\mathbb{Z}$.

18.2 The case where y has a pole at $t=\infty$ and $c_{\infty,0}\neq 1/2 \cdots$ (1)

Proposition 18.7. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1$ and there exists a rational solution of type B such that y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$. Moreover, assume that case (1) occurs in Proposition 1.16. Then, one of the following occurs:*

(1) $\alpha_4+\alpha_5\in\mathbb{Z}$, (2) $\alpha_3=0$, (3) $\alpha_2+\alpha_3=0$, (4) $\alpha_2+\alpha_3-1/2\in\mathbb{Z}$,
(5) $\alpha_3-1/2\in\mathbb{Z}$, (6) $\alpha_0+\alpha_1\in\mathbb{Z}$, (7) $2\alpha_3+\alpha_4+\alpha_5\in\mathbb{Z}$.

Proof. We can assume that $\alpha_3 \neq 0$. $s_3(x,y,z,w)$ is then a rational solution of type B of $D_5^{(1)}(\alpha_0,\alpha_1,\alpha_2+\alpha_3,-\alpha_3,\alpha_4+\alpha_3,\alpha_5+\alpha_3)$ such that $s_3(y)$ has a pole at $t=\infty$ and all of $s_3(x,z,w)$ are holomorphic at $t=\infty$. Moreover, for $s_3(x,y,z,w)$,

$$c_{\infty,0} = \frac{\alpha_5}{-\alpha_4+\alpha_5} + \frac{\alpha_3}{-2\alpha_3(-\alpha_4+\alpha_5)/(\alpha_4+\alpha_5)} = \frac{1}{2}.$$

Thus, from Proposition 18.6, we can prove the proposition. □

18.3 The case where y has a pole at $t=\infty$ and $c_{\infty,0} \neq 1/2 \cdots$ (2)

18.3.1 The case where x,y,z,w are all holomorphic at $t=0$

Proposition 18.8. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, $-\alpha_0+\alpha_1=0$, $-\alpha_4+\alpha_5=1$ and there exists a rational solution of type B such that both* (1) *and* (2) *hold:*
(1) *y has a pole at $t=\infty$ and x,z,w are holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$ and case* (2) *occurs in Proposition 1.16.*
(2) *x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}+d_{0,0}=0$, $b_{0,0}=0$.*
Either of the following then occurs: (1) $\alpha_4=0$, (2) $2\alpha_3+\alpha_4+\alpha_5\in 2\mathbb{Z}$.

Proof. From Proposition 4.6 and 4.10, it follows that

$$h_{\infty,0}-h_{0,0} = \frac{1}{4}(2\alpha_3+\alpha_4+\alpha_5)^2 \in \mathbb{Z}. \tag{18.5}$$

$\pi_2(x,y,z,w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_5,\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_0)$ such that all of $\pi_2(x,y,z,w)$ are holomorphic at $t=0$ and $b_{0,0}+d_{0,0}=0$, $b_{0,0}=0$. Moreover, either of the following occurs:
(i) $\pi_2(w)$ has a pole at $t=\infty$ and all of $\pi_2(x,y,z)$ are holomorphic at $t=\infty$,
(ii) both of $\pi_2(y,w)$ have a pole at $t=\infty$ and both of $\pi_2(x,z)$ are holomorphic at $t=\infty$.

If case (i) occurs, it follows from Proposition 1.26 that $\alpha_4=0$.

If case (ii) occurs, case (2) occurs in Proposition 1.41. From Proposition 4.9 and 4.10, it follows that

$$h_{\infty,0}-h_{0,0} = \frac{1}{4}(2\alpha_3+\alpha_4+\alpha_5)^2 - \frac{1}{2}(2\alpha_3+\alpha_4+\alpha_5) \in \mathbb{Z}. \tag{18.6}$$

We then observe from (18.5) and (18.6) that $2\alpha_3+\alpha_4+\alpha_5\in 2\mathbb{Z}$. □

Proposition 18.9. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, $-\alpha_0+\alpha_1=0$, $-\alpha_4+\alpha_5=1$ and there exists a rational solution of type B such that the following hold:*
(1) *y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$ and case* (2) *occurs in Proposition 1.16,*
(2) *x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}+d_{0,0}=0$, $b_{0,0}\neq 0$.*
Either of the following then occurs: (1) $\alpha_4=0$, (2) $\alpha_0+\alpha_1-1\in 2\mathbb{Z}$.

Proof. From Proposition 4.6 and 4.10, it follows that

$$h_{\infty,0} - h_{0,0} = \frac{1}{4}(2\alpha_3 + \alpha_4 + \alpha_5)^2 + \alpha_2(\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5) \in \mathbb{Z}. \tag{18.7}$$

$\pi_2(x, y, z, w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0)$ such that z has a pole at $t = 0$ and x, y, w are all holomorphic at $t = 0$. Moreover, either of the following occurs:
(i) $\pi_2(w)$ has a pole at $t = \infty$ and all of $\pi_2(x, y, z)$ are holomorphic at $t = \infty$,
(ii) both of $\pi_2(y, w)$ have a pole at $t = \infty$ and both of $\pi_2(x, z)$ are holomorphic at $t = \infty$.

If case (i) occurs, it follows from Proposition 1.26 that $\alpha_4 = 0$.

If case (ii) occurs, case (2) occurs in Proposition 1.41. From Proposition 4.9 and 4.12, it follows that

$$h_{\infty,0} - h_{0,0} = \frac{1}{4}(2\alpha_4 + \alpha_4 + \alpha_5)^2 - \frac{1}{2}(2\alpha_3 + \alpha_4 + \alpha_5) - \alpha_2(\alpha_0 + \alpha_1 + \alpha_2) \in \mathbb{Z}. \tag{18.8}$$

Thus, we observe from (18.7) and (18.8) that $\alpha_0 + \alpha_1 - 1 \in 2\mathbb{Z}$. □

Proposition 18.10. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, $-\alpha_0 + \alpha_1 = 0$, $-\alpha_4 + \alpha_5 = 1$ and there exists a rational solution of type B such that the following hold:*
(1) y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} \neq 1/2$ and case (2) occurs in Proposition 1.16,
(2) x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5$, $b_{0,0} \neq 0$.
Then, $\alpha_0 + \alpha_1 \in \mathbb{Z}$.

Proof. We can assume that $\alpha_0 \neq 0$. $s_0(x, y, z, w)$ is then a rational solution of type B of $D_5^{(1)}(-\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4, \alpha_5)$ such that $s_0(y)$ has a pole at $t = \infty$ and all of $s_0(x, z, w)$ are holomorphic at $t = \infty$. Moreover, all of $s_0(x, y, z, w)$ are holomorphic at $t = 0$. Therefore, it follows from the residue theorem that

$$-(-\alpha_0) + \alpha_1 = -\mathrm{Res}_{t=\infty} s_0(x) \in \mathbb{Z}.$$

□

Proposition 18.11. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, $-\alpha_0 + \alpha_1 = 0$, $-\alpha_4 + \alpha_5 = 1$ and there exists a rational solution of type B such that the following hold:*
(1) y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} \neq 1/2$ and case (2) occurs in Proposition 1.16,
(2) x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} + d_{0,0} = -\alpha_4 + \alpha_5$, $b_{0,0} = 0$.
Then, $\alpha_4 + \alpha_5 \in \mathbb{Z}$.

Proof. We can assume that $\alpha_4 \neq 0$. $s_1\pi_2(x,y,z,w)$ is then a rational solution of type B of $D_5^{(1)}(\alpha_5, -\alpha_4, \alpha_3+\alpha_4, \alpha_2, \alpha_1, \alpha_0)$ such that both of $s_1\pi_2(y,w)$ have a pole at $t=\infty$ and both of $s_1\pi_2(x,z)$ are holomorphic at $t=\infty$ and case (1) occurs in Proposition 1.41. Moreover, all of $s_1\pi_2(x,y,z,w)$ are holomorphic at $t=0$. Therefore, it follows from the residue theorem that

$$-\alpha_5 + (-\alpha_4) = -\mathrm{Res}_{t=\infty} s_1\pi_2(x) \in \mathbb{Z}.$$

□

18.3.2 The case where z has a pole at $t=0$

Proposition 18.12. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=0$, $-\alpha_4+\alpha_5=1$ and there exists a rational solution of type B such that the following hold:*
(1) *y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$ and case* (2) *occurs in Proposition 1.16,*
(2) *z has a pole of order n $(n\geq 1)$ at $t=0$ and x,y,w are all holomorphic at $t=0$.*
Either of the following then occurs: (1) $\alpha_4=0$, (2) $\alpha_4+\alpha_5\in 2\mathbb{Z}$.

Proof. We can assume that $\alpha_4\neq 0$. Then, let us consider the following two cases:
(i) z has a pole of order n $(n\geq 2)$ at $t=0$ and x,y,w are all holomorphic at $t=0$,
(ii) z has a pole of order one at $t=0$ and x,y,w are all holomorphic at $t=0$.

We first treat case (i). If case (ii) occurs, the proposition can be proved in the same way.

From Proposition 4.6 and 4.12, it follows that

$$h_{\infty,0} - h_{0,0} = \frac{1}{4}(\alpha_4+\alpha_5)^2 \in \mathbb{Z}. \tag{18.9}$$

$\pi_2(x,y,z,w)$ is a rational solution of type B of $D_5^{(1)}(\alpha_5,\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_0)$ such that both of $\pi_2(y,w)$ have a pole at $t=\infty$ and both of $\pi_2(x,z)$ are holomorphic at $t=\infty$ and case (2) occurs in Proposition 1.41. Moreover, both $\pi_2(y,w)$ have a pole of order $n-1$ at $t=0$ and both of $\pi_2(x,z)$ are holomorphic at $t=0$. Then, from Proposition 4.9 and 4.14, it follows that

$$h_{\infty,0} - h_{0,0} = \frac{1}{4}(\alpha_4+\alpha_5)^2 - \frac{1}{2}(\alpha_4+\alpha_5) \in \mathbb{Z}. \tag{18.10}$$

Therefore, we observe from (18.9) and (18.10) that $\alpha_4+\alpha_5\in 2\mathbb{Z}$. □

18.3.3 The case where y,w have a pole at $t=0$

Proposition 18.13. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=0$, $-\alpha_4+\alpha_5=1$ and there exists a rational solution of type B such that the following hold:*

(1) *y has a pole at $t=\infty$ and x, z, w are all holomorphic at $t=\infty$ and $c_{\infty,0} \neq 1/2$ and case* (2) *occurs in Proposition 1.16,*
(2) *y, w both have a pole of order n ($n \geq 1$) at $t=0$ and x, z are both holomorphic at $t=0$.*
One of the following then occurs: (1) $\alpha_2 = 0$, (2) $2\alpha_3 + \alpha_4 + \alpha_5 = 0$, (3) $\alpha_4 + \alpha_5 = 0$.

Proof. We can assume that $\alpha_2 \neq 0$. It then follows from Corollary 2.19 that $s_2(x, y, z, w)$ is a rational solution of $D_5^{(1)}(\alpha_0+\alpha_2, \alpha_1+\alpha_2, -\alpha_2, \alpha_3+\alpha_2, \alpha_4, \alpha_5)$ such that all of $s_2(x, y, z, w)$ are holomorphic at $t=0$ and $a_{0,0} = c_{0,0} = 1/2$. The proposition follows from Lemma 2.20. ☐

18.3.4 summary

Proposition 18.14. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0 + \alpha_1 = 0$, $-\alpha_4 + \alpha_5 = 1$ and there exists a rational solution of type B such that y has a pole at $t=\infty$ and x, z, w are all holomorphic at $t=\infty$ and case* (2) *occurs in Proposition 1.16. One of the following then occurs:*

(1) $\alpha_2 = 0$, (2) $\alpha_4 = 0$, (3) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, (4) $\alpha_4 + \alpha_5 \in \mathbb{Z}$, (5) $2\alpha_3 + \alpha_4 + \alpha_5 \in \mathbb{Z}$.

18.4 The case where y has a pole at $t=\infty$ and $c_{\infty,0} \neq 1/2 \cdots$ (3)

Proposition 18.15. *If $-\alpha_0 + \alpha_1 = 0$ and $-\alpha_4 + \alpha_5 = 1$, then, for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, there exists no rational solution of type B such that y has a pole at $t=\infty$ and x, z, w are all holomorphic at $t=\infty$ and case* (3) *occurs in Proposition 1.16.*

Proof. If case (3) occurs in Proposition 1.16, it follows that $\alpha_4 = \alpha_5 = 0$, which is impossible. ☐

18.5 Summary of cases (0), (1), (2) and (3)

Proposition 18.16. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0 + \alpha_1 = 0$, $-\alpha_4 + \alpha_5 = 1$ and there exists a rational solution of type B such that y has a pole at $t=\infty$ and x, z, w are all holomorphic at $t=\infty$. One of the following then occurs:*

(1) $\alpha_2 = 0$, (2) $\alpha_3 = 0$, (3) $\alpha_2 + \alpha_3 = 0$, (4) $\alpha_2 - 1/2 \in \mathbb{Z}$,
(5) $\alpha_3 - 1/2 \in \mathbb{Z}$, (6) $\alpha_2 + \alpha_3 - 1/2 \in \mathbb{Z}$, (7) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, (8) $2\alpha_3 + \alpha_4 + \alpha_5 \in \mathbb{Z}$,
(9) $\alpha_4 + \alpha_5 \in \mathbb{Z}$.

18.6 The case where w has a pole at $t = \infty$

Proposition 18.17. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, $-\alpha_0 + \alpha_1 = 0$, $-\alpha_4 + \alpha_5 = 1$ and there exists a rational solution of type B such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$. Then, $-\alpha_0 + \alpha_1 = 0$, $-\alpha_4 + \alpha_5 = 1$, $\alpha_1 = 0$.*

Proof. The proposition follows from Proposition 1.26. □

18.7 The case where y, z have a pole at $t = \infty$

Proposition 18.18. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, $-\alpha_0 + \alpha_1 = 0$, $-\alpha_4 + \alpha_5 = 1$ and there exists a rational solution of type B such that y, z both have a pole at $t = \infty$ and x, w are both holomorphic at $t = \infty$. Then, $\alpha_4 + \alpha_5 = 0$.*

Proof. The proposition follows from Proposition 1.33. □

18.8 The case where y, w have a pole at $t = \infty$

Proposition 18.19. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, $-\alpha_0 + \alpha_1 = 0$, $-\alpha_4 + \alpha_5 = 1$ and there exists a rational solution of type B such that y, w both have a pole of order one at $t = \infty$ and x, z are both holomorphic at $t = \infty$. Then, $\alpha_0 = 0$.*

Proof. The proposition follows from Proposition 1.41. □

By s_2, we can obtain the following proposition:

Proposition 18.20. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, $-\alpha_0 + \alpha_1 = 0$, $-\alpha_4 + \alpha_5 = 1$ and there exists a rational solution of type B such that y, w have a pole of order n, $(n \geq 1)$ at $t = \infty$ and x, z are both holomorphic at $t = \infty$. One of the following then occurs:*

(1) $\alpha_1 = 0$, (2) $\alpha_2 = 0$, (3) $\alpha_3 = 0$, (4) $\alpha_2 + \alpha_3 = 0$,
(5) $\alpha_2 - 1/2 \in \mathbb{Z}$, (6) $\alpha_3 - 1/2 \in \mathbb{Z}$, (7) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, (8) $\alpha_2 + \alpha_3 - 1/2 \in \mathbb{Z}$,
(9) $\alpha_4 + \alpha_5 \in \mathbb{Z}$, (10) $2\alpha_3 + \alpha_4 + \alpha_5 \in \mathbb{Z}$.

18.9 Summary

Proposition 18.21. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, $-\alpha_0 + \alpha_1 = 0$, $-\alpha_4 + \alpha_5 = 1$ and there exists a rational solution of type B. One of the following then occurs:*

(1) $\alpha_2 = 0$, (2) $\alpha_3 = 0$, (3) $\alpha_2 + \alpha_3 = 0$, (4) $\alpha_2 - 1/2 \in \mathbb{Z}$,
(5) $\alpha_2 + \alpha_3 - 1/2 \in \mathbb{Z}$, (6) $\alpha_3 - 1/2 \in \mathbb{Z}$, (7) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, (8) $\alpha_4 + \alpha_5 \in \mathbb{Z}$,
(9) $2\alpha_3 + \alpha_4 + \alpha_5 \in \mathbb{Z}$.

Corollary 18.22. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1$ *and there exists a rational solution of type B. Moreover, assume that the parameters satisfy no one of the ten conditions in Theorem 9.5. By some Bäcklund transformations, the parameters can then be transformed so that one of the following occurs:*

(1) $-\alpha_0+\alpha_1=0,-\alpha_4+\alpha_5=1,\alpha_1=0,$ (2) $-\alpha_0+\alpha_1=0,-\alpha_4+\alpha_5=1,\alpha_2=0,$
(3) $-\alpha_0+\alpha_1=0,-\alpha_4+\alpha_5=1,\alpha_3=0,$ (4) $-\alpha_0+\alpha_1=0,-\alpha_4+\alpha_5=1,\alpha_2=1/2,$
(5) $-\alpha_0+\alpha_1=0,-\alpha_4+\alpha_5=1,\alpha_3=1/2,$ (6) $-\alpha_0+\alpha_1=0,-\alpha_4+\alpha_5=1,\alpha_4=0.$

19 Rational solutions for the standard form II (2)

19.1 The case where $-\alpha_0+\alpha_1=0,-\alpha_4+\alpha_5=1,\alpha_1=0$

Proposition 19.1. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=0,-\alpha_4+\alpha_5=1,\alpha_1=0$ *and there exists a rational solution of type B. One of the following then occurs:*
(1) $\alpha_2=0$, (2) $\alpha_3=0$, (3) $\alpha_4+\alpha_5\in\mathbb{Z}$, (4) $2\alpha_3+\alpha_4+\alpha_5\in\mathbb{Z}$, (5) $\alpha_3-1/2\in\mathbb{Z}$.

Proof. From the discussion in Section 18, we have only to consider the following cases:
(1) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$ and x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5,\ b_{0,0}\neq 0$,
(2) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$ and case (1) occurs in Proposition 1.16,
(3) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$ and case (2) occurs in Proposition 1.16 and x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5,\ b_{0,0}\neq 0$,
(4) w has a pole at $t=\infty$ and x,y,z are all holomorphic at $t=\infty$,
(5) y,w both have a pole at $t=\infty$ and x,z are both holomorphic at $t=\infty$.

Using the formula $a_{\infty,-1}-a_{0,-1}\in\mathbb{Z}$ and $h_{\infty,0}-h_{0,0}\in\mathbb{Z}$ in the same way, we can prove the proposition. □

Corollary 19.2. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1,\ \alpha_1=0$ *and there exists a rational solution of type B. Moreover, assume that the parameters satisfy no one of the ten conditions in Theorem 9.5. By some Bäcklund transformations, the parameters can then be transformed so that either of the following occurs:*
(1) $\alpha_0=\alpha_1=\alpha_4=0,\ \alpha_5=1,$ (2) $-\alpha_0+\alpha_1=0,\ \alpha_2=\alpha_4=0,\ \alpha_5=1,$
(3) $-\alpha_0+\alpha_1=0,\ \alpha_2=1/2,\ \alpha_4=0,\ \alpha_5=1.$

19.2 The case where $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1,\ \alpha_2=0$

Proposition 19.3. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1,\ \alpha_2=0$

and there exists a rational solution of type B. One of the following then occurs:
(1) $\alpha_3 = 0$, (2) $\alpha_3 - 1/2 \in \mathbb{Z}$, (3) $\alpha_4 + \alpha_5 \in \mathbb{Z}$, (4) $2\alpha_3 + \alpha_4 + \alpha_5 \in \mathbb{Z}$.

Proof. From the discussion in Section 18, we have only to consider the following cases:
(1) y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} = 1/2$ and z has a pole of order n $(n \geq 1)$ and x, y, w are all holomorphic at $t = 0$,
(2) y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} = 1/2$ and y, w both have a pole of order n $(n \geq 1)$ at $t = 0$ and x, z are both holomorphic at $t = 0$,
(3) y has a pole at $t = \infty$ and x, z, w are holomorphic at $t = \infty$ and $c_{\infty,0} \neq 1/2$ and case (2) occurs in Proposition 1.16 and y, w both have a pole of order n $(n \geq 1)$ at $t = 0$ and x, z are both holomorphic at $t = 0$,
(4) y, w both have a pole of order n $(n \geq 2)$, at $t = \infty$ and x, z are both holomorphic at $t = \infty$.

Using the formula $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$ and $h_{\infty,0} - h_{0,0} \in \mathbb{Z}$ in the same way, we can prove the proposition. □

Corollary 19.4. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_2 = 0$ *and there exists a rational solution of type B. Moreover, assume that the parameters satisfy no one of the ten conditions in Theorem 9.5. The parameters can then be transformed so that one of the following occurs:*

(1) $\alpha_0 = \alpha_1 = \alpha_4 = 0,\ \alpha_5 = 1$, (2) $-\alpha_0 + \alpha_1 = 0, -\alpha_4 + \alpha_5 = 1, \alpha_2 = \alpha_3 = 0$,
(3) $-\alpha_0 + \alpha_1 = 0, \alpha_2 = \alpha_4 = 0, \alpha_5 = 1$, (4) $-\alpha_0 + \alpha_1 = 0, -\alpha_4 + \alpha_5 = 1, \alpha_2 = \alpha_3 = 1/2$.

19.3 The case where $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_3 = 0$

Proposition 19.5. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_3 = 0$ *and there exists a rational solution of type B. One of the following then occurs:*
(1) $\alpha_1 = 0$, (2) $\alpha_2 = 0$, (3) $\alpha_2 - 1/2 \in \mathbb{Z}$, (4) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, (5) $\alpha_4 + \alpha_5 \in \mathbb{Z}$.

Proof. From the discussion in Section 18, we have only to consider the following cases:
(1) y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} \neq 1/2$ and case (1) occurs in Proposition 1.16,
(2) y, w both have a pole of order n $(n \geq 2)$ at $t = \infty$ and x, z are both holomorphic at $t = \infty$.

Using the formula $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$ and $h_{\infty,0} - h_{0,0} \in \mathbb{Z}$ in the same way, we can prove the proposition. □

Corollary 19.6. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_3 = 0$ *and there exists a rational solution of type B. Moreover, assume that the parameters satisfy*

no one of the ten conditions in Theorem 9.5. By some Bäcklund transformations, the parameters can then be transformed so that one of the following occurs:

(1) $-\alpha_0 + \alpha_1 = 0, \alpha_2 = 0, \alpha_4 = 0, \alpha_5 = 1$, (2) $-\alpha_0 + \alpha_1 = 0, -\alpha_4 + \alpha_5 = 1, \alpha_2 = \alpha_3 = 0$,
(3) $\alpha_0 = \alpha_1 = \alpha_4 = 0,\ \alpha_5 = 1$, (4) $-\alpha_0 + \alpha_1 = 0, -\alpha_4 + \alpha_5 = 1, \alpha_2 = \alpha_3 = 1/2$.

19.4 The case where $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_2 = 1/2$

Proposition 19.7. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_2 = 1/2$ and there exists a rational solution of type B. One of the following then occurs:*
(1) $\alpha_3 - 1/2 \in \mathbb{Z}$, (2) $\alpha_3 \in \mathbb{Z}$, (3) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, (4) $\alpha_4 + \alpha_5 \in \mathbb{Z}$.

Proof. From the discussion in Section 18, we have only to consider the following cases:
(1) y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} \neq 1/2$ and x, y, z, w are all holomorphic at $t = 0$ and $b_{0,0} = d_{0,0} = 0$,
(2) y, w both have a pole of order n $(n \ge 2)$ at $t = \infty$ and x, z are both holomorphic at $t = \infty$.

Using the formula $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$ and $h_{\infty,0} - h_{0,0} \in \mathbb{Z}$ in the same way, we can prove the proposition. □

Corollary 19.8. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_2 = 1/2$ and there exists a rational solution of type B. Moreover, assume that the parameters satisfy no one of the ten conditions in Theorem 9.5. By some Bäcklund transformations, the parameters can then be transformed so that one of the following occurs:*
(1) $-\alpha_0 + \alpha_1 = 0, \alpha_2 = 1/2, \alpha_4 = 0, \alpha_5 = 1$, (2) $-\alpha_0 + \alpha_1 = 0, -\alpha_4 + \alpha_5 = 1, \alpha_2 = \alpha_3 = 1/2$.

19.5 The case where $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_3 = 1/2$

Proposition 19.9. *Suppose that for $D_5^{(1)}(\alpha_j)_{0\le j\le 5}$, $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_3 = 1/2$ and there exists a rational solution of type B. One of the following then occurs:*
(1) $\alpha_2 - 1/2 \in \mathbb{Z}$, (2) $\alpha_2 \in \mathbb{Z}$, (3) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, (4) $\alpha_4 + \alpha_5 \in \mathbb{Z}$.

Proof. From the discussion in Section 18, we have only to consider the following cases:
(1) y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} = 1/2$ and z has a pole of order n at $t = 0$ and x, y, w are all holomorphic at $t = 0$,
(2) y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$ and $c_{\infty,0} \neq 1/2$ and case (1) occurs in Proposition 1.16,
(3) y, w both have a pole of order n $(n \ge 2)$ at $t = \infty$ and x, z are both holomorphic at $t = \infty$.

Using the formula $a_{\infty,-1} - a_{0,-1} \in \mathbb{Z}$ and $h_{\infty,0} - h_{0,0} \in \mathbb{Z}$ in the same way, we can prove the proposition. □

Corollary 19.10. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1,\ \alpha_3=1/2$ *and there exists a rational solution of type B. Moreover, assume that the parameters satisfy no one of the ten conditions in Theorem 9.5. By some Bäcklund transformations, the parameters can then be transformed so that either of the following occurs:*
(1) $-\alpha_0+\alpha_1=0, -\alpha_4+\alpha_5=1, \alpha_2=\alpha_3=1/2$, (2) $-\alpha_0+\alpha_1=0, \alpha_2=1/2, \alpha_4=0,\ \alpha_5=1$,

19.6 The case where $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1,\ \alpha_4=0$

Proposition 19.11. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1,\ \alpha_4=0$ *and there exists a rational solution of type B. One of the following then occurs:*
(1) $\alpha_2=0$, (2) $\alpha_2-1/2\in\mathbb{Z}$, (3) $\alpha_3-1/2\in\mathbb{Z}$, (4) $\alpha_0+\alpha_1\in\mathbb{Z}$, (5) $\alpha_3\in\mathbb{Z}$.

Proof. From the discussion in Section 18, we have only to consider the following cases:
(1) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}=1/2$ and x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5,\ b_{0,0}=0,\ c_{0,0}=\alpha_5/(-\alpha_4+\alpha_5)\neq 1/2$,
(2) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$ and case (1) occurs in Proposition 1.16,
(3) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$ and case (2) occurs in Proposition 1.16 and x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}+d_{0,0}=0$,
(4) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$ and case (2) occurs in Proposition 1.16 and x,y,z,w are all holomorphic at $t=0$ and $b_{0,0}+d_{0,0}=-\alpha_4+\alpha_5,\ b_{0,0}=0$,
(5) y has a pole at $t=\infty$ and x,z,w are all holomorphic at $t=\infty$ and $c_{\infty,0}\neq 1/2$ and case (2) occurs in Proposition 1.16 and z has a pole of order n, $(n\geq 1)$ at $t=0$ and x,y,w are all holomorphic at $t=0$,
(6) y,w both have a pole of order n $(n\geq 2)$ at $t=\infty$ and x,z are both holomorphic at $t=\infty$.

Using the formula $a_{\infty,-1}-a_{0,-1}\in\mathbb{Z}$ and $h_{\infty,0}-h_{0,0}\in\mathbb{Z}$ in the same way, we can prove the proposition. □

Corollary 19.12. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=0,\ -\alpha_4+\alpha_5=1,\ \alpha_4=0$ *and there exists a rational solution of type B. Moreover, assume that the parameters satisfy no one of the ten conditions in Theorem 9.5. By some Bäcklund transformations, the parameters can then be transformed so that one of the following occurs:*

(1) $-\alpha_0+\alpha_1=0,\ \alpha_2=\alpha_4=0,\ \alpha_5=1$, (2) $\alpha_0=\alpha_1=\alpha_4=0,\ \alpha_5=1$,
(3) $-\alpha_0+\alpha_1=0,\ \alpha_2=1/2,\ \alpha_4=0,\ \alpha_5=1$.

19.7 Summary

Proposition 19.13. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=0, -\alpha_4+\alpha_5=1$ *and there exists a rational solution of type B. Moreover, assume that the parameters satisfy no one of the ten conditions in Theorem 9.5. By some Bäcklund transformations, the parameters can then be transformed so that one of the following occurs:*

(1) $\alpha_0=\alpha_1=\alpha_4=0,\ \alpha_5=1,$ (2) $-\alpha_0+\alpha_1=0, \alpha_2=\alpha_4=0, \alpha_5=1,$
(3) $-\alpha_0+\alpha_1=0, -\alpha_4+\alpha_5=1, \alpha_2=\alpha_3=0,$ (4) $-\alpha_0+\alpha_1=0, -\alpha_4+\alpha_5=1, \alpha_2=\alpha_3=1/2,$
(5) $-\alpha_0+\alpha_1=0, \alpha_2=1/2, \alpha_4=0, \alpha_5=1.$

20 Rational solutions for the standard form II (3)

In this section, we treat the five cases in Proposition 19.13. We can prove the following propositions and corollaries in the same way as in the previous section.

20.1 The case where $\alpha_0=\alpha_1=\alpha_4=0,\ \alpha_5=1$

Proposition 20.1. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $\alpha_0=\alpha_1=\alpha_4=0,\ \alpha_5=1$ *and there exists a rational solution of type B. Either of the following then occurs:*
(1) $\alpha_2\in\mathbb{Z},\ \alpha_3\in\mathbb{Z},$ (2) $\alpha_2-1/2\in\mathbb{Z},\ \alpha_3-1/2\in\mathbb{Z}.$

Corollary 20.2. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $\alpha_0=\alpha_1=\alpha_4=0,\ \alpha_5=1$ *and there exists a rational solution of type B. Moreover, assume that the parameters satisfy no one of the ten conditions in Theorem 9.5. By some Bäcklund transformations, the parameters can then be so transformed so that* $\alpha_0=\alpha_1=\alpha_2=\alpha_3=\alpha_4=0,\ \alpha_5=1.$

20.2 The case where $-\alpha_0+\alpha_1=0,\ \alpha_2=\alpha_4=0,\ \alpha_5=1$

Proposition 20.3. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=0,\ \alpha_2=\alpha_4=0,\ \alpha_5=1$ *and there exists a rational solution of type B. Either of the following then occurs:*
(1) $\alpha_3\in\mathbb{Z},$ (2) $\alpha_3-1/2\in\mathbb{Z}.$

Corollary 20.4. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$, $-\alpha_0+\alpha_1=0,\ \alpha_2=\alpha_4=0,\ \alpha_5=1$ *and there exists a rational solution of type B. Moreover, assume that the parameters satisfy no one of the ten conditions in Theorem 9.5. By some Bäcklund transformations, the parameters can then be transformed so that* $\alpha_0=\alpha_1=\alpha_2=\alpha_3=\alpha_4=0,\ \alpha_5=1.$

20.3 The case where $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_2 = \alpha_3 = 0$

Proposition 20.5. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_2 = \alpha_3 = 0$ *and there exists a rational solution of type B. Either of the following then occurs:* (1) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, (2) $\alpha_4 - 1/2 \in \mathbb{Z}$.

Corollary 20.6. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, $-\alpha_0+\alpha_1 = 0,\ -\alpha_4+\alpha_5 = 1,\ \alpha_2 = \alpha_3 = 0$ *and there exists a rational solution of type B. Moreover, assume that the parameters satisfy no of the ten conditions in Theorem 9.5. By some Bäcklund transformations, the parameters can then be transformed so that* $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0,\ \alpha_5 = 1,$

20.4 The case where $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_2 = \alpha_3 = 1/2$

Proposition 20.7. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_2 = \alpha_3 = 1/2$ *and there exists a rational solution of type B. Then,*

$$-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_2 = \alpha_3 = 1/2,\ \alpha_4 + \alpha_5 \in \mathbb{Z}.$$

Corollary 20.8. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, $-\alpha_0 + \alpha_1 = 0,\ -\alpha_4 + \alpha_5 = 1,\ \alpha_2 = \alpha_3 = 1/2$ *and there exists a rational solution of type B. The parameters then satisfy one of the conditions in Theorem 9.5 and by some Bäcklund transformations, the parameters can be transformed that* $-\alpha_0 + \alpha_1 = -\alpha_4 + \alpha_5 = 0$.

20.5 The case where $-\alpha_0 + \alpha_1 = 0,\ \alpha_2 = 1/2,\ \alpha_4 = 0,\ \alpha_5 = 1$

Proposition 20.9. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, $-\alpha_0 + \alpha_1 = 0,\ \alpha_2 = 1/2,\ \alpha_4 = 0,\ \alpha_5 = 1$ *and there exists a rational solution of type B. Then,*

$$-\alpha_0 + \alpha_1 = 0,\ \alpha_2 = 1/2,\ \alpha_4 = 0,\ \alpha_5 = 1,\ \alpha_0 + \alpha_1 \in \mathbb{Z}.$$

Corollary 20.10. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, $-\alpha_0+\alpha_1 = 0,\ \alpha_2 = 1/2,\ \alpha_4 = 0,\ \alpha_5 = 1$ *and there exists a rational solution of type B. The parameters then satisfy one of the conditions in Theorem 9.5 and by some Bäcklund transformations, the parameters can be transformed that* $-\alpha_0 + \alpha_1 = -\alpha_4 + \alpha_5 = 0$.

20.6 Summary

Proposition 20.11. *Suppose that for* $D_5^{(1)}(\alpha_j)_{0 \le j \le 5}$, $-\alpha_0 + \alpha_1 = 0, -\alpha_4 + \alpha_5 = 1$ *and there exists a rational solution of type B. Moreover, assume that the parameters satisfy no one of the ten conditions in Theorem 9.5. By some Bäcklund transformations, the parameters can then be transformed so that* $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0,\ \alpha_5 = 1.$

21 Rational solutions for the standard form II (4)

In this section, we prove that for $D_5^{(1)}(0,0,0,0,0,1)$, there exists no rational solution of type B.

Proposition 21.1. *Suppose that for $D_5^{(1)}(0,0,0,0,0,1)$, there exists a rational solution of type B. Then,*

$$h_{\infty,0} = \begin{cases} 0 & \text{if } y, z \text{ both have a pole at } t = \infty, \\ 1/4 & \text{otherwise}. \end{cases}$$

Proof. It can be proved by direct calculation. ☐

Proposition 21.2. *For $D_5^{(1)}(0,0,0,0,0,1)$, there exists no rational solution of type B such that y has a pole of order one at $t=\infty$ and z has a pole of order n $(n \geq 1)$ at $t = \infty$.*

Proof. The proposition follows from 1.33. ☐

Proposition 21.3. *For $D_5^{(1)}(0,0,0,0,0,1)$, there exists no rational solution of type B.*

Proof. Suppose that $D_5^{(1)}(0,0,0,0,0,1)$ has a rational solution of type B. From the discussions in Section 4, considering $h_{\infty,0} - h_{0,0} \in \mathbb{Z}$, we can assume that y, w both have a pole at $t = 0$ and x, z are both holomorphic at $t = 0$.

$s_3 s_5(x,y,z,w)$ is then a rational solution of $D_5^{(1)}(0,0,1,-1,1,0)$ such that y, w both have a pole at $t=0$ and x, z are both holomorphic at $t = 0$, which contradicts Lemma 2.21. ☐

22 Summary for the standard form II

Theorem 22.1. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, $-\alpha_0 + \alpha_1 = 0, -\alpha_4 + \alpha_5 = 1$ and there exists a rational solutions of type B. The parameters then satisfy one of the conditions in Theorem 9.5.*

Proof. The theorem follows from Propositions 20.11 and 21.3. ☐

23 The main theorem for type B

Theorem 23.1. *Suppose that for $D_5^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a rational solution of type B. By some Bäcklund transformations, the parameters and solution can then be transformed so that either of the following occurs:*

(b-1) $-\alpha_0+\alpha_1=-\alpha_4+\alpha_5=0$ *and* $(x,y,z,w)=(1/2,-t/2,1/2,0)$,
(b-2) $(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=(1/2,1/2,0,\alpha_3,-\alpha_3,-\alpha_3)$ *and*

$$(x,y,z,w)=(1/2,-t/2+b,1/2,d),$$

where b, d *are both arbitrary complex numbers and satisfy* $b+d=0$.

Moreover, $D_5^{(1)}(\alpha_j)_{0\leq j\leq 5}$ *has a rational solution of type B if and only if the parameters satisfy one of the following conditions:*

(1)	$-\alpha_0+\alpha_1\in\mathbb{Z}$,	$-\alpha_4+\alpha_5\in\mathbb{Z}$,	$-\alpha_0+\alpha_1\equiv-\alpha_4+\alpha_5$	mod 2,
(2)	$-\alpha_0+\alpha_1\in\mathbb{Z}$,	$-\alpha_4-\alpha_5\in\mathbb{Z}$,	$-\alpha_0+\alpha_1\equiv-\alpha_4-\alpha_5$	mod 2,
(3)	$-\alpha_0+\alpha_1\in\mathbb{Z}$,	$-\alpha_0-\alpha_1\in\mathbb{Z}$,	$-\alpha_0+\alpha_1\not\equiv-\alpha_0-\alpha_1$	mod 2,
(4)	$-\alpha_0-\alpha_1\in\mathbb{Z}$,	$-\alpha_4+\alpha_5\in\mathbb{Z}$,	$-\alpha_0-\alpha_1\equiv-\alpha_4+\alpha_5$	mod 2,
(5)	$-\alpha_0-\alpha_1\in\mathbb{Z}$,	$-\alpha_4-\alpha_5\in\mathbb{Z}$,	$-\alpha_0-\alpha_1\equiv-\alpha_4-\alpha_5$	mod 2,
(6)	$-\alpha_4-\alpha_5\in\mathbb{Z}$,	$-\alpha_4+\alpha_5\in\mathbb{Z}$,	$-\alpha_4-\alpha_5\not\equiv-\alpha_4+\alpha_5$	mod 2,
(7)	$-\alpha_0+\alpha_1\in\mathbb{Z}$,	$-2\alpha_3-\alpha_4-\alpha_5\in\mathbb{Z}$,	$-\alpha_0+\alpha_1\equiv-2\alpha_3-\alpha_4-\alpha_5$	mod 2,
(8)	$-\alpha_0-\alpha_1\in\mathbb{Z}$,	$-2\alpha_3-\alpha_4-\alpha_5\in\mathbb{Z}$,	$-\alpha_0+\alpha_1\equiv-2\alpha_3-\alpha_4-\alpha_5$	mod 2,
(9)	$-\alpha_0-\alpha_1-2\alpha_2\in\mathbb{Z}$,	$-\alpha_4+\alpha_5\in\mathbb{Z}$,	$-\alpha_0-\alpha_1-2\alpha_2\equiv-\alpha_4+\alpha_5$	mod 2,
(10)	$-\alpha_0-\alpha_1-2\alpha_2\in\mathbb{Z}$,	$-\alpha_4-\alpha_5\in\mathbb{Z}$,	$-\alpha_0-\alpha_1-2\alpha_2\equiv-\alpha_4-\alpha_5$	mod 2.

Proof. The theorem follows from Theorems 17.1 and 22.1. □

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